Financial Derivatives & Risk Management
Professor J.P. Singh
Department of Management Studies
Indian Institute of Technology Roorkee
Lecture 47
Black–Scholes Model Contd.

Now, I come to the Black Scholes Model. I have discussed the pricing of options in the discrete framework using a discrete random walk with time in discrete steps and price, also, in discontinuous jumps i.e. both the time and the stochastic variable were taken as discontinuous. Now, I relax both these assumptions. I take up the pricing of options in a continuous time & continuous variable framework. This is the celebrated Black Scholes model. I start with the assumptions of the model.

Assumptions of the Black Scholes model

(i) The stock price follows the lognormal process with constant mean return and volatility.
(ii) The short selling of securities with full use of proceeds is permitted.
(iii) There are no transactions costs or taxes. All securities are perfectly divisible.
(iv) There are no dividends during the life of the derivative.
(v) There are no riskless arbitrage opportunities.
(vi) Security trading is continuous.
(vii) The risk-free rate of interest, r, is constant and the same for all maturities.

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The fundamental assumption is that the stock price follows a lognormal process with constant mean return and volatility. Thus,

$$\ln S_T \xrightarrow{distribution} N\left[\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right]$$

This is in the real world. In the risk neutral world we shall replace the expected return (μ) by the riskfree rate (r). However, volatility (σ) will remain unchanged due to the result of the Girsanov's theorem. The distribution remains the same.

The short selling of securities with full use of the proceeds is allowed. Short selling means that an investor can borrow the asset and sell it in the market in anticipation of a price decline so that when the price falls he can buy the asset and replenish it to the original owner.

There are no transaction costs or taxes. That means there is no friction in the market. The markets are efficient.

Further, there is no distortion created due to differential taxes. In many countries, it is the practice that the capital gains tax rate is slightly lower than the regular tax rates, as a result of which a distortion is created between capital income and dividend income. These distortions are not contemplated by the model.

There are no dividends during the life of the derivative. This is another fundamental assumption. It may be recalled that this model values European derivatives i.e. those that can be exercised only at maturity.

There are no riskless arbitrage opportunities. In other words, we are assuming a fully efficient market so that arbitrage will neutralize any price differences between assets of identical risk-return characteristics instantaneously.

The discreteness of security units is ignored. Trading prices are assumed continuous, that is, bid and offer orders can be placed at any arbitrary prices. There is no mandated tick size.

The riskfree rate is constant and is the same for all maturities.

<u>Derivation of the Black Scholes PDE</u>

Consider a derivative $C \equiv C(S,t)$, which is a function of the instantaneous price, $S_t \equiv S$ of an underlying asset S and also an explicit function of time. S follows the stochastic process:

$$dS = \mu S dt + \sigma S dW_t \qquad (1)$$

We assume that C(S,t) satisfies all the requirements for the application of Ito's Lemma i.e. it is continuous, at least twice differentiable etc.

By Ito's Lemma, given a function $G \equiv G(x_t,t)$ which is a continuous and at least twice differentiable function in its arguments $x_t \equiv x,t$ where x follows the stochastic process:

$dx = a(x,t)dt + b(x,t)dW_t$ we have, $dG = \left(a\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial x^2} \right)dt + b\frac{\partial G}{\partial x}dW_t$ (2)

Applying Ito's Lemma to C(S,t) with S following the process (1) i.e. $dS = \mu S dt + \sigma S dW$, we obtain:

$$dC = \left(\mu S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} \right)dt + \sigma\frac{\partial C}{\partial S}dW_t \qquad (3)$$

From this expression we see that the change in the derivative price in an infinitesimal time increment dt can be segregated into terms viz.

(i) the deterministic drift that does not contain any randomness and which is proportional to the time length dt given by the first term of the above expression viz. $\left(\mu S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} \right)dt$; and

(ii) the diffusion term $\sigma \frac{\partial C}{\partial S} dW_t$, which encapsulates the randomness manifest due to the BM increment dW_t. Clearly, the fluctuations in the derivative price and hence, the risk arises due to this term.

Construction of the riskless portfolio

It is seen from the above, that the randomness and hence, the risk in the derivative price arise from the BM increment i.e. the term

$$\sigma \frac{\partial C}{\partial S} dW_t \tag{4}$$

It is also seen from the stock price model $dS=\mu Sdt+\sigma SdW_t$ that the diffusion term in the stock price (that encapsulates the randomness and hence, the risk) is

$$\sigma SdW_t \tag{5}$$

per unit of the stock.

By comparing (4) & (5), it follows that, if we construct a portfolio Π consisting of:

(i) one unit of the derivative; and

(ii) $-\frac{\partial C}{\partial S}$ units of the underlying stock,

then the diffusion terms and hence, the randomness will cancel each other and we shall have created a riskless portfolio. This can be explicitly seen as follows:

$$d\Pi = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW_t - \frac{\partial C}{\partial S}\left(\mu Sdt + \sigma SdW_t \right)$$

$$= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt \tag{6}$$

Minus represents that the position would be opposite to the position of the derivative. If the derivative is long the stock would be short and if the derivative is short then stock would be long. The value of Π at the point of construction is equal to C which is the price of one unit of the derivative (long, say) less the amount financed by short selling of $\frac{\partial C}{\partial S}$ units of the stock yielding S per unit.

The assumption embedded here is that over this interval dt what we are talking about, $-\frac{\partial C}{\partial S}$ does not change. In other words, the portfolio composition does not change. Because if $-\frac{\partial C}{\partial S}$ changes, the portfolio, composition changes because the portfolio has been defined that way. The portfolio has been defined in terms of 1: $-\frac{\partial C}{\partial S}$.Thus, if the slope $-\frac{\partial C}{\partial S}$ changes then the portfolio composition will also change. But we are assuming that time period dt is so small that $-\frac{\partial C}{\partial S}$ remains constant over the small period of time. And therefore dΠ is given by eq. (6).

Since, the expression for dΠ does not contain any random term, it is riskless and will generate the riskfree rate of return over dt i.e.

$$\frac{1}{\Pi}\frac{d\Pi}{dt} = r \text{ or } d\Pi = r\Pi dt = r\left(C - \frac{\partial C}{\partial S}S\right)dt \qquad (7)$$

since Π consists of one unit of derivative (say long) which costs C and $-\frac{\partial C}{\partial S}$ units of stock, each of which costs S. Please note the negative sign because the positions are opposite.

Equating (6) & (7), we get the Black Scholes PDE as:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \qquad (8)$$

Interesting to note that the expected return on the stock does not figure in the PDE, rather it is the riskfree return that appears, as was the case in the binomial model.

Boundary conditions

$$For\ a\ call\ option: C\left(S_T, T\right) = \max\left(S_T - K, 0\right);\ For\ a\ put\ option: P\left(S_T, T\right) = \max\left(K - S_T, 0\right)$$

The terminal payoff from a European call is max(S_T-K,0) and that from a European put is max(K-S_T,0), which will constitute the appropriate terminal conditions.

BS solutions

$$c = S_0\mathbf{N}\left(d_1\right) - Ke^{-rT}\mathbf{N}\left(d_2\right);\ p = Ke^{-rT}\mathbf{N}\left(-d_2\right) - S_0\mathbf{N}\left(-d_1\right)$$

$$d_1 = \frac{\ln\left(S_0/K\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}};\ d_2 = \frac{\ln\left(S_0/K\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

where $if\ Z \xrightarrow{distribution} N\left(0,1\right)\ then\ P\left(Z \le z\right) = \mathbf{N}\left(z\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} d\omega\, \mathbf{exp}\left(-\frac{\omega^2}{2}\right)$ i.e. $\mathbf{N}\left(z\right)$ is the cumulative standard normal distribution function.

It needs to be emphasized here that while calculating the Black Scholes value of the derivative, care should be taken to use the riskfree rate in computing values of d_1 and d_2 and not using the expected return on the stock in the real world.

This is clear from the Black Scholes PDE $\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC=0$ which involves the riskfree rate r and not the expected return μ. Recall that even in binomial pricing, we arrived at the price as the discounted expectation of the payoff function $c=E_Q[f(S_T)]$, that expectation being worked out using risk neutral probabilities Q. Further, those risk neutral probabilities were $q=\frac{e^{rT}-d}{u-d}$ i.e. dependent on the riskfree rate and not the expected return.

If we carefully analyse the derivation of the BS eq, we find that the real world return gets eliminated, when we construct the hedge consisting of the derivative and the hedge (stock).

<u>Black Scholes: Probability of call exercise in the risk-neutral world</u>

We know that the payoff from a European call at maturity is $\max(S_T-K,0)$. Hence, it will be exercised only if $S_T>K$. Hence, we need to find out $P(S_T>K) = P(\ln S_T>\ln K)$ since ln (x) is a one-one monotonically increasing function of x. But, $\ln S_T \xrightarrow{distribution} N\left[\ln S_0+\left(r-\frac{1}{2}\sigma^2\right)T,\sigma^2 T\right]$ in the risk-neutral world. Hence, we have, on standardizing the normal variate,

$$P(\ln S_T>\ln K)=1-P(\ln S_T<\ln K)=1-P\left\{Z<\frac{\ln K-\left[\ln S_0+\left(r-\frac{1}{2}\sigma^2\right)T\right]}{\sigma\sqrt{T}}\right\}$$

$$=1-\mathbf{N}\left\{\frac{\ln K-\left[\ln S_0+\left(r-\frac{1}{2}\sigma^2\right)T\right]}{\sigma\sqrt{T}}\right\}=\mathbf{N}\left\{-\left[\frac{\ln K-\left[\ln S_0+\left(r-\frac{1}{2}\sigma^2\right)T\right]}{\sigma\sqrt{T}}\right]\right\}=\mathbf{N}(d_2)$$

Remember we are working in the risk neutral world so are we will not have μ here, we will have r, the risk free rate. Because we are working in the risk neutral world the return will be risk free rate and not the expected stock return in the in the real market.

<u>Delta of Black Scholes call</u>

We have, $c=S\mathbf{N}(d_1)-K^{-rT}\mathbf{N}(d_2);\quad \Delta=\frac{\partial c}{\partial S}=\mathbf{N}(d_1)+S\mathbf{N}'(d_1)-Ke^{-rT}\mathbf{N}'(d_2)$. Now,

$$\mathbf{N}'(d_1)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}\frac{\partial d_1}{\partial S}\mathbf{;}\quad \frac{\partial d_1}{\partial S}=\frac{\partial}{\partial S}\left\{\frac{\mathbf{ln}\ S-\mathbf{ln}\ K+\left[\left(r-\frac{1}{2}\sigma^2\right)T\right]}{\sigma\sqrt{T}}\right\}=\frac{1}{\sigma S\sqrt{T}}$$ so that

$$\mathbf{N}'(d_1)=\frac{1}{\sigma S\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}$$ whence $S\mathbf{N}'(d_1)=\frac{1}{\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}$. Proceeding similarly,

$$\mathbf{N}'(d_2)=\frac{1}{\sigma S\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2}$$ whence $Ke^{-rT}\mathbf{N}'(d_2)=\frac{Ke^{-rT}}{\sigma S\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2}$. Hence,

$$S\mathbf{N}'(d_1)-Ke^{-rT}\mathbf{N}'(d_2)=\frac{1}{\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}d_1^2}-\frac{Ke^{-rT}}{S}e^{-\frac{1}{2}d_2^2}\right)=\frac{1}{\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}d_1^2}-e^{\mathbf{ln}\left(\frac{K}{S}\right)-rT-\frac{1}{2}d_2^2}\right)=0$$

where the last step follows due to

$$\mathbf{ln}\left(\frac{K}{S}\right)-rT-\frac{1}{2}d_2^2=\mathbf{ln}\left(\frac{K}{S}\right)-rT-\frac{1}{2}\left(d_1-\sigma\sqrt{T}\right)^2=\mathbf{ln}\left(\frac{K}{S}\right)-rT-\frac{1}{2}d_1^2-\frac{1}{2}\sigma^2T+d_1\sigma\sqrt{T}$$

$$=-\left[\mathbf{ln}\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T\right]-\frac{1}{2}d_1^2+d_1\sigma\sqrt{T}=-d_1\sigma\sqrt{T}-\frac{1}{2}d_1^2+d_1\sigma\sqrt{T}=-\frac{1}{2}d_1^2.$$ Therefore, we have, $\Delta=\frac{\partial c}{\partial S}=\mathbf{N}(d_1)$.