Financial Derivatives & Risk Management
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Lecture 46 - Girsanov's Theorem; Black Scholes Model

We have discussed earlier that in the case of American call on a non-dividend paying stock, it is never optimal to early exercise the American call. If an investor owns an American call on a stock and the stock is not expected to pay any dividend during the lifetime of the option, then it is never optimal that the option should be exercised early. The important premise in this contention is that the stock should be non-dividend paying. Remember it is an American option, so it can be exercised early i.e. before maturity but by virtue of arguments, we established that it is not optimal to exercise the option early.

We, now, relax this assumption and examine the optimality of early exercise of American call on a stock that is expected to pay dividend during the lifetime of the option.

Early exercise of American calls on dividend paying stock

We are going to investigate now whether, in the case of American calls on a dividend paying stock, it can ever be optimal to early exercise the option.

We first establish that when examining the optimality of early exercise of American calls on dividend paying stocks, it can, if at all, only be optimal to exercise at a time immediately before the stock goes ex-dividend.

For simplicity, we assume that there is only one dividend payment, say D, during the life of the call option. We assume that the stock goes ex-dividend at $t=T_d$ in (0,T) where T is the maturity of the option.

Now, if an option holder exercises the option any time before dividend date, he gets the stock for the exercise price and also the dividend on the dividend date if he holds the stock till then.

However, if he delays the exercise until just before the ex-dividend date, then he still gets the stock at the cost of the exercise price and also the dividend. Hence, in so far as the cost of getting the stock is concerned and the dividend thereon, it does not really matter whether he exercises the option just before the dividend date or much earlier. But by deferring the exercise to just before the ex-dividend date, he gets the following advantages:

(i) the cash outflow on account of exercise price gets deferred resulting in saving of interest cost; and
(ii) if the investor exercises the option just before the ex-dividend date, then, in the event that the stock price falls below the exercise price, he can decide upon avoiding the exercise of the option and acquiring the stock in the market, a possibility that he foregoes if he exercises the option earlier.

Further, if he plans to sell the stock before the ex-dividend date, he can get a better profit by selling the unexercised option, compared to exercising the option and then selling the stock.

Hence, we shall consider the possibility of early exercise only just before the ex-dividend date i.e. when the stock goes ex-dividend. In other words, it is sufficient for us to investigate the optimality of early exercise at point just before $t=T_d$ where T_d is the ex-dividend date i.e. it is not necessary for us to examine the viability of this proposition at a date earlier to the ex-dividend date.

In view of the above, we need to compare the following situations:

(i) Exercise of the American call immediately before the stock goes ex-dividend i.e. at $t=T_d$; and
(ii) Carrying the call unexercised.

In case (i), since we are considering only one dividend, D, for which the stock goes ex-dividend at $t=T_d$, the option will, obviously, be exercised only if $S_{T_d} > K$ (because if $S_{T_d} < K$, the option exercise will make no sense as the payoff in that case will be zero) and will yield the payoff $S_{T_d} - K$.

In case (ii) i.e. if the option is not exercised, the stock price will drop down to $S_{T_d} - D$ immediately after the stock goes ex-dividend. The stock price will fall by the amount of dividend immediately after it goes ex-dividend. With this stock price, i.e. after the stock has gone ex-dividend, the lower bound on the option price will be $S_{T_d} - D - Ke^{-r(T-T_d)}$. Note that the lower bound on the call option price at any arbitrary time t in (0,T) during the life of the option is given by: Current Stock Price at t – Present Value of Exercise Price at t.

Thus, if $S_{T_d} - D - Ke^{-r(T-T_d)} > S_{T_d} - K$ or $D < K\left(1-e^{-r(T-T_d)}\right) \approx Kr\left(T-T_d\right)$, it is surely not optimal to exercise the option because the lower bound on the option price post ex-dividend is exceeding the payoff from the option before ex-dividend so that the least value of the unexercised option is more than the payoff on exercise.

On the other hand, if $D>K\{1-\exp[-r(T-T_d)]\}$ it may or may not be optimal to exercise the call. If D is greater than $K\{1-\exp[-r(T-T_d)]\}$, the above analysis does not conclusively establish that it is optimal to exercise the call option.

However, for any reasonable assumption about the stochastic process followed by the stock price, it can be shown that it is always optimal to exercise at time T_d for a sufficiently high value of S_{T_d}. The > inequality will tend to be satisfied when the ex-dividend date is fairly close to the maturity of the option (i.e., $T–T_d$ is small) and the dividend D is large.

We have concluded that if the stock pays no dividend, then it is certainly not optimal to early exercise an American call, because the LHS of $D < K\left(1-e^{-r(T-T_d)}\right) \approx Kr\left(T-T_d\right)$ would be zero and hence, the inequality will invariably be satisfied. So if D is zero then it is surely not optimal to exercise the call early, which we have already established by other grounds. So the current argument is consistent with the earlier approach. But if the dividend is large enough and this period is small, in other words the payment of dividend is close to the maturity of the call then there could be a possibility that early exercise could be optimal.

Now we come to another interesting result. The interesting result is called Girsanov's Theorem. What Girsanov's Theorem says, although it is a very technical theorem, we shall talk about, we shall leave out the mathematical nuances. We shall talk about the philosophy of the theorem and in context of our pricing, what it says is that see we have been talking about real life probabilities, we have been talking about risk neutral probabilities.

So it is interesting to explore the inter-relationship between them and Girsanov's theorem gives us that inter-relationship. The inter-relationship says that when we move from one world to the other, that is when we move from the real world to the risk neutral-world or vice versa, the expected return changes. The drift changes, however the volatility remains unchanged. The volatility, the underlying volatility, if you move from the risk neutral-world to the real world or vice versa remains unaffected. Of course, the rates of return and trends or the drift terms do undergo a change.

Girsanov's Theorem

Consider a single step binomial tree of length T in the risk neutral world. The stock price at t=0 is S_0 and at the end of the single time period of length T, it can jump to either uS_0 with probability q_u or dS_0 with probability q_d=1-q_u.Thus, $S_T = \begin{cases} uS_0 \ \ with \ \ probability \ \ q_u \\ dS_0 \ \ with \ \ probability \ \ q_d = 1 - q_u \end{cases}$. Then, we have, the expected stock price at t=T i.e. at the end of the jump $E(S_T) = quS_0 + (1\text{-}q)dS_0$. Now, if r is the riskfree rate of return, then the expected stock price at t=T is $E(S_T) = S_0 e^{rT}$ whence we get $q = \frac{e^{rT} - d}{u - d}$.

Now, if the stock, at t=T, jumps to uS_0, the probability for which is q, the percentage change in price is $\%_u = \frac{uS_0 - S_0}{S_0} = u - 1$ and if the stock price falls to dS_0, the probability for which is 1-q, the percentage change is $\%_d$ = d-1. Thus, the variance of the percentage change in price is:

$$Var(\%) = E(\%^2) - [E(\%)]^2 = q(\%_u)^2 + (1-q)(\%_d)^2 - [q(\%_u) + (1-q)(\%_d)]^2$$
$$= q(u-1)^2 + (1-q)(d-1)^2 - [q(u-1) + (1-q)(d-1)]^2$$
$$= qu^2 - 2qu + q + (1-q)d^2 - 2(1-q)d + (1-q) - [qu + (1-q)d - q - (1-q)]^2$$
$$= qu^2 - 2q(u-d) + (1-q)d^2 - 2d + 1 - [qu + (1-q)d - 1]^2$$
$$= qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 = q(u^2 - d^2) + d^2 - [q(u-d) + d]^2$$
$$Using \ q = \frac{e^{rT} - d}{u - d}, \ we \ get \ Var(\%) = (e^{rT} - d)(u + d) + d^2 - e^{2rT} = e^{rT}(u + d) - ud - e^{2rT}$$

Now, if we define σ^2 as the variance of the percentage change in stock price per unit time, then we have, $e^{rT}(u + d) - ud - e^{2rT} = \sigma^2 T$. A possible solution to this equation due to Cox &

Rubinstein upto first order in T is given by: $u = e^{\sigma\sqrt{T}}$; $d = e^{-\sigma\sqrt{T}}$ as shown below: $ud = 1$; $u + d = \left(1 + \sigma\sqrt{T} + \frac{1}{2}\sigma^2 T + ...\right) + \left(1 - \sigma\sqrt{T} + \frac{1}{2}\sigma^2 T + ...\right) = 2 + \sigma^2 T + ...$ so that $e^{rT}(u+d) - ud - e^{2rT} = (1 + rT + ...)(2 + \sigma^2 T + ...) - 1 - (1 + 2rT + ...) = \sigma^2 T + O(T^2)$.'

Proceeding similarly, we can show that the variance of the percentage change in stock price over time T in the real world yields a similar expression viz. $Var(\%)_{real} = pu^2 + (1-p)d^2 - \left[pu + (1-p)d\right]^2 = p(u^2 - d^2) + d^2 - \left[p(u-d) + d\right]^2$ where, however, p is the real-world probability of an upswing, given by, $p = \frac{e^{\mu T} - d}{u - d}$ and μ is the real-world expected return. Using this value of p, we get:

$$Var(\%)_{real} = \left(e^{\mu T} - d\right)(u+d) + d^2 - e^{2\mu T} = e^{\mu T}(u+d) - ud - e^{2\mu T}$$

If we substitute $u = e^{\sigma\sqrt{T}}$; $d = e^{-\sigma\sqrt{T}}$ in this equation, we find, on ignoring higher powers of T, that the expression equals $\sigma^2 T$.

And what we end up with is basically that the expression for u and d of the drift rates in the risk-neutral world r or real world μ. Although the probabilities q in the risk-neutral world and p in the real world depend upon these drift rates, the jump sizes (volatilities) are independent of these rates. In either case, we find that the same u and d satisfy the requisite equations.

The above analysis shows that when we move from the risk neutral world to the real world or vice versa, the expected return on the stock changes, but its volatility remains the same, at least in the limit as $T \to 0$. This is an illustration of Girsanov's theorem, a celebrated result in stochastic finance.

The drift rate does change on transition, u and d and hence, the volatility does not change. The jumps do not change.

Putting it more generally, when we move from a world with one set of risk preferences to a world with another set of risk preferences, the expected growth rate in variable changes but the volatility remain the same.