Financial Derivatives & Risk Management
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Option Pricing: Binomial Model Contd.

Risk-neutral pricing

(i) The expected return on stock price (calculated with reference to q-probabilities) equals the riskfree rate.
(ii) Now, if the stock price (a risky asset) is giving an expected return of the riskless asset, then there are two issues:

Implications

(A) All investors are accepting the same return irrespective of the risk profile of the asset;

There would be a number of risky assets in the market, of which this particular stock could be one. And yet, this symptomatic risky asset is yielding an expected return to all investors that is independent of its riskiness. That can only happen if the investors are entirely indifferent to the risk profile of the investments. In other words, the risk of the investment does not constitute a decision criterion for them or that the investors are risk-neutral.

(B) That return is the riskfree return.

From the principle of arbitrage this return had to be the riskfree return. Because if any asset provided a higher return in the market, notwithstanding the fact that it had a higher risk, investors would invariably invest in the risky asset because it gives them higher return and they were not concerned about risk. So they invest in the risky asset which enhances the demand for that asset and reduces the demand for the riskfree asset. As a result of this demand imbalance, the prices of the risky asset escalate and that of the riskfree asset goes down and the returns tend to equalize. Thus, the lowest return would be sustainable in the market. Now, because the riskfree asset has the least risk, it is logical that it would be providing the lowest return. It follows that all investors would get the riskfree return.

The bottom-line of the above analysis is that the q-probabilities reflect the probabilities of stock price movements in a risk neutral world i.e. in a world where risk has no significance to each and every investor and return is the only investment criterion. Hence, these probabilities are also called risk neutral probabilities.

Thus, the price of a derivative is the present value of the expected payoff from the derivative on maturity, the expectation being calculated with reference to risk neutral probabilities.

Binomial pricing: call option

Now let us look at an example of the Binomial Pricing Model. We consider the case of a European call option. Now, the payoff of the European call on maturity is $f(S_T)=\max(S_T-K,0)$. Hence, its price at an arbitrary t=0 would be, using risk neutral pricing:

$$c = e^{-rT} E_Q\left[f\left(S_T\right)\right] = e^{-rT} E_Q\left[\max\left(S_T - K, 0\right)\right]$$

Now in the case of one-step binomial model, the stock price can take only two values, $S_u = uS_0$ with probability q_u and $S_d = dS_0$ with probability q_d. Now in this particular model it has to be, that the strike price K will lie between S_u and S_d. If the exercise price $K < S_d$, then the option will invariably be exercised. In fact, it would take the character of the stock itself. So it really does not carry any worth in terms of being an option. Its payoff is simply the payoff of the stock shifted by the amount of exercise price i.e. S_T-K and so its value will simply be the current stock price less present value of exercise price i.e. S_0-Kexp(-rT).

If $K > S_u$, the option will never be exercised, because the market price of stock can never exceed S_u. So the option will never be exercised and hence, it is worthless and will have a value of zero.

So we must have, in a realistic situation, $S_u > K > S_d$ i.e. the exercise price must lie between the upper value of the stock and lower value of the stock at maturity. Hence, we have the following:

$$c = e^{-rT} E_Q\left[f\left(S_T\right)\right] = e^{-rT} E_Q\left[\mathbf{max}\left(S_T - K, 0\right)\right]$$

$$\textit{In the one step binomial } \mathbf{mod}\textit{el}: S_T = \begin{bmatrix} S_u \textit{ with prob } q_u \\ S_d \textit{ with prob } q_d \end{bmatrix}$$

$$\textit{Now, in this } \mathbf{mod}\textit{el}, \textit{ it must be } S_u > K > S_d$$

$$\textit{Thus, option payoff } f\left(S_T\right) = \begin{bmatrix} S_u - K \textit{ with prob } q_u \\ 0 \textit{ with prob } q_d \end{bmatrix}$$

Thus, $S_u > K > S_d$. Now, if $S_T = S_d$ (the probability of which is q_d) then option payoff $f(S_T)=0$, the probability of which is, therefore, q_d. And, if $S_T = S_u$ (the probability of which is q_u) then option payoff $f(S_T)=S_u$-K, the probability of which is, therefore, q_u. Therefore, the call price of today is:

$$c = e^{-rT}\left[q_u\left(S_u - K\right)\right] = e^{-rT}\left(q_u S_u - q_u K\right) \qquad (1)$$

Now, the option will be exercised only if $S_T > K$ i.e. only if $S_T = S_u$, the probability of which is q_u, then the option holder will pay the exercise price K i.e. if the option is exercised the option holder incurs a cash outflow of K i.e. a cash flow of –K. Thus, by taking a long option position, the option holder incurs a cash flow of –K with probability q_u. Of course, if the stock finishes at S_d, the probability of which is q_d, there will be no cash flows on account of exercise price since the option will not be exercised. Hence, expected value of cash outflow of the option holder:

$$\begin{aligned} &\textit{Expected value of cash outflow under the option} \\ &= \mathbf{Pr}\textit{ob of non-exercise of option} \times 0\left(\textit{because no exercise price will be paid}\right) \\ &+ \mathbf{Pr}\textit{ob of exercise of option} \times \textit{exercise price} = q_d 0 - q_u K = -q_u K \end{aligned}$$

E(cash outflow) $= q_d{*}0 - q_u{*}K = -q_u{*}K$

The minus sign signifies that it is an outflow from the perspective of the option holder.

However, if the holder exercises the option (the probability of which is q_u), he gets one unit of the stock that has a market price of S_u. Thus, the long position in the option entitles the holder to a cash inflow of S_u with probability q_u. But if the stock finishes at S_d, the probability of which is q_d, he will not exercise the option and hence, he will not receive the stock and so no cash inflow will materialize. Hence,

$$\begin{aligned}
&\textit{Expected value of cash } \mathbf{inf}\textit{low under the option} \\
&= \mathbf{Pr}\textit{ob of non-exercise of option} \times 0 \left(\textit{because no stock will be delivered}\right) \\
&+ \mathbf{Pr}\textit{ob of exercise of option} \times \textit{stock price if option is exercised} = q_d 0 + q_u S_u = q_u S_u \\
&\textit{and } e^{-rT} q_u S_u = PV \textit{ of } \mathbf{exp} \textit{ value of cash } \mathbf{inf}\textit{low under the option} \\
&\textit{Thus } c = e^{-rT} q_u S_u - e^{-rT} q_u K = PV \textit{ of } \mathbf{exp} \textit{ value of net cash flow}
\end{aligned}$$

E(cash inflow) $=q_d*0+q_u*S_u=+q_u*S_u$

Thus, E(present value of net cash flow) = $e^{-rT}*q_u*(S_u-K)$ (2)

Comparing (1) & (2), we clearly find that the value of the option equals the discounted expected value of the net cash flow to the holder or equivalently, the discounted expected cash inflow less the discounted expected cash outflow $e^{-rT}*q_u*S_u- e^{-rT}*q_uK$.

Present value of the expected cash inflow in terms of the receipt of stock minus present value of the expected cash outflow in terms of the payment of exercise money. And that is nothing but the present value of the expected net cash flow.

We have, thus, concluded that the value of the call is the present value of the expected net cash flow under the option.

Two period binomial model: European option

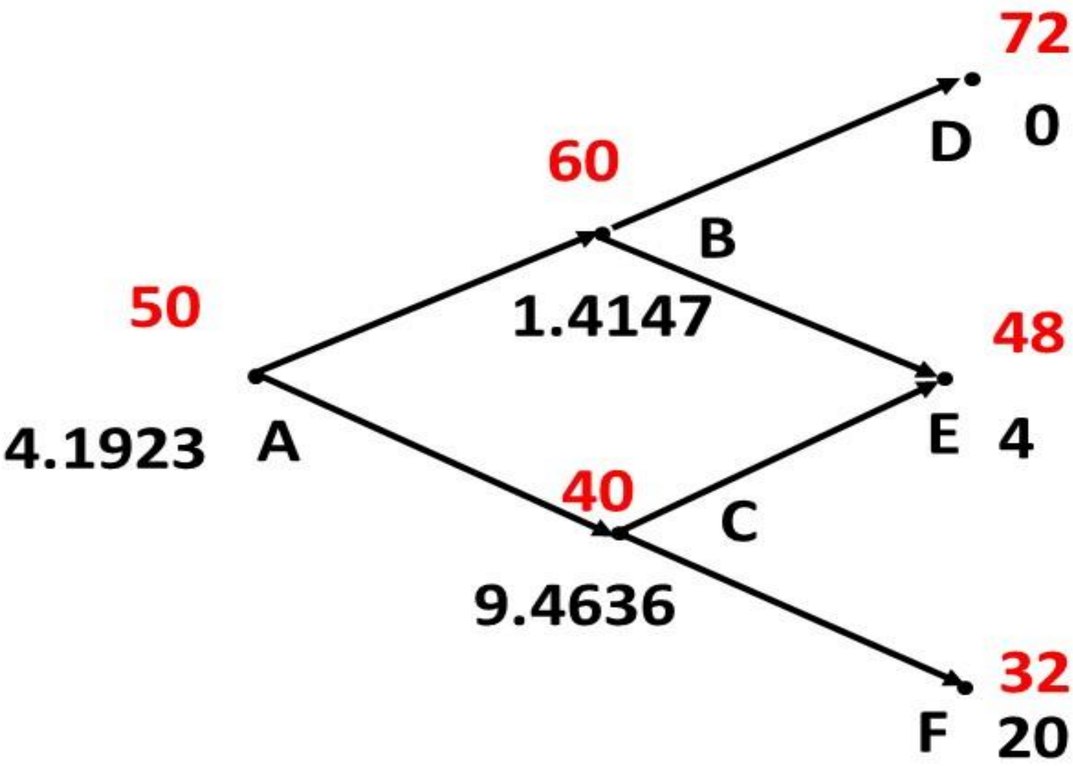
Example 1

Consider a 2-year European put with a strike price of 52 on a stock whose current price is 50. In each time step (of one year) the stock price either moves up by 20% or moves down by 20%. Let the risk-free interest rate be 5%. Calculate the current price of the option using a twostep binomial model.

Solution

In the given problem: $u = 1.20;\ d = 0.80;\ T = 1 \textit{ year};\ r = 5\%$ and these parameters are constant throughout the tree.

$$\begin{aligned}
&\textit{Stock prices at } \mathbf{var}\textit{ious nodes}: \\
&S_{uu} = 72;\ S_{ud} = S_{du} = 48;\ S_{dd} = 32;\ S_u = 60;\ S_d = 40;\ S_0 = 50 \\
&\textit{Claim values at } \mathbf{var}\textit{ious nodes}: f\left(S_T\right) = \mathbf{max}\left(K - S_T, 0\right) \\
&f_{uu} = \mathbf{max}\left(52-72,0\right) = 0;\ f_{ud} = \mathbf{max}\left(52-48,0\right) = 4;\ f_{dd} = \mathbf{max}\left(52-32,0\right) = 20 \\
&\textit{Risk neutral probabilities}: q_u = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05\times1} - 0.80}{1.20 - 0.80} = 0.6282
\end{aligned}$$



$$f_u = e^{-rT}\left[qf_{uu} + (1-q)\, f_{ud}\right] = 0.9512(0.6282\times 0 + 0.3718\times 4) = 1.4147$$
$$f_d = e^{-rT}\left[qf_{ud} + (1-q)\, f_{dd}\right] = 0.9512(0.6282\times 4 + 0.3718\times 20) = 9.4636$$
$$f_0 = e^{-rT}\left[qf_{u} + (1-q)\, f_{d}\right] = 0.9512(0.6282\times 1.4147 + 0.3718\times 9.4636) = 4.1923$$

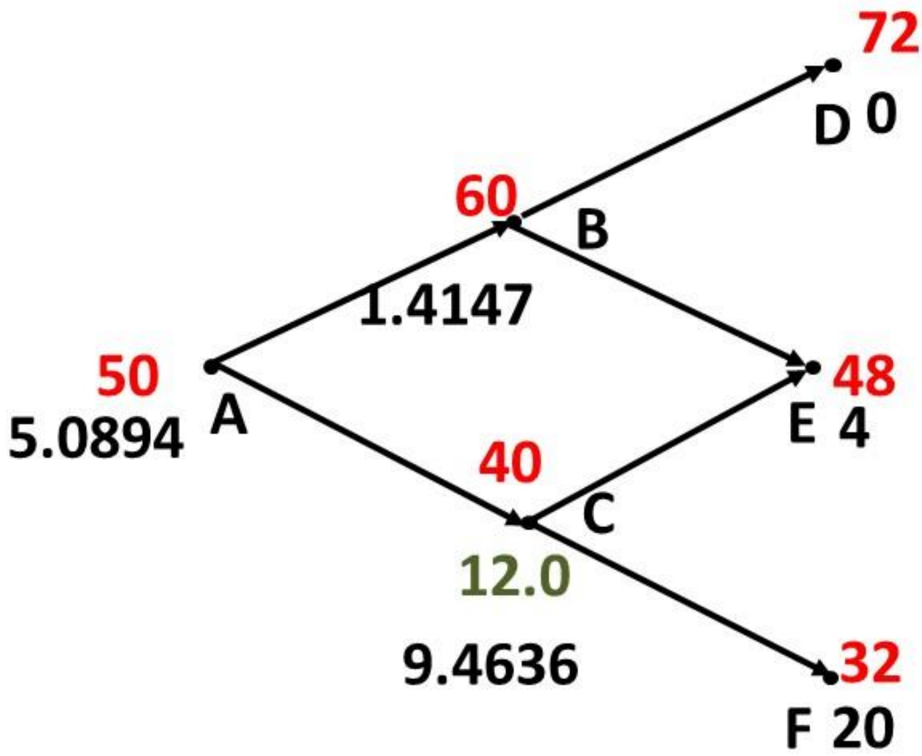
Two period binomial model: American option

Consider a 2-year European put with a strike price of 52 on a stock whose current price is 50. In each time step (of one year) the stock price either moves up by 20% or moves down by 20%. Let the risk-free interest rate be 5%. Calculate the current price of the option using a twostep binomial model.

Solution

The data in this problem is the same as in the previous example. However, here the instrument is an American put and not a European put. Hence, we need to understand how the American-ness of the option manifests itself in the pricing formulation. Recall that an American option can be exercised at any time upto maturity whereas the European option can be exercised only at maturity, Hence, we need to assess the value addition or value generation due to this possibility of early exercise.

In other words, we have to examine whether it would be profitable for the investor to exercise the option early i.e. at the nodes B or C or even A because he has the right to exercise it at these nodes as per the contract. Hence, he can make a choice whether to exercise the option at A or B or C and let it continue unexercised till maturity i.e. D, E &F.



So we have to examine the possibility of at what point in time, the investor may find it optimal to exercise the option. That is the difference between the valuation that we have done for the European put and the American put. Let us now work through the problem.

As earlier, we have $u = 1.20;\ d = 0.80;\ T = 1\ year;\ r = 5\%$ and these parameters are constant throughout the tree.

$$\textit{Stock prices at various nodes}:$$
$$S_{uu} = 72;\ S_{ud} = S_{du} = 48;\ S_{dd} = 32;\ \ S_u = 60;\ S_d = 40;\ S_0 = 50$$
$$\textit{Claim values at various nodes}:\ f\left(S_T\right) = \mathbf{max}\left(K - S_T, 0\right)$$
$$f_{uu} = \mathbf{max}\left(52 - 72, 0\right) = 0;\ f_{ud} = \mathbf{max}\left(52 - 48, 0\right) = 4;\ f_{dd} = \mathbf{max}\left(52 - 32, 0\right) = 20$$
$$\textit{Risk neutral probabilities}:\ q_u = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05\times 1} - 0.80}{1.20 - 0.80} = 0.6282$$

<u>Early exercise at B</u>

Using this data, we find that the theoretical value of the option at the node B works out to 1.4147 as below:

$$f_u = e^{-rT}\left[qf_{uu} + \left(1 - q\right) f_{ud}\right] = 0.9512\left(0.6282\times 0 + 0.3718\times 4\right) = 1.4147$$

Now, we examine the payoff if the investor decides to early exercise the option at B. The stock price at B is projected at 60 while the put's exercise price is 52. Hence, the payoff on early exercise at B will be max(K-S_B,0) =0. Hence, the investor will not get any payoff if he early exercises the put at B. He is better off carrying the option unexercised, since then it commands a theoretical value of 1.4147.

$$f_u = 1.4147;\ f_u^{early\ exercise} = \mathbf{max}\left(K - S_u, 0\right) = \mathbf{max}\left(52 - 60, 0\right) = 0$$

<u>Early exercise at C</u>

We find that the theoretical value of the option at the node C works out to 9.4636 as below:

$$f_d = e^{-rT}\left[qf_{ud} + \left(1-q\right) f_{dd}\right] = 0.9512\left(0.6282 \times 4 + 0.3718 \times 20\right) = 9.4636$$

Now, we examine the payoff if the investor decides to early exercise the option at C. The stock price at C is projected at 40 while the put's exercise price is 52. Hence, the payoff on early exercise at C will be max(K-S_C,0) =12. Hence, the investor will get a payoff of 12 if he early exercises the put at C. If he does not exercise the option at C but carries it unexercised, its value at C is 9.4636. Hence, he is better off exercising the option at the node C rather than carrying it forward unexercised.

Now, because, he will exercise the option at C if the stock reaches the node C and he will derive a payoff from the option at C itself of 12, the option will be worth 12 at C and NOT 9.4636. Thus, when we work out the value of the option at A by rolling the tree backwards, the put value at C will need to be considered at 12 and not at 9.4636.

$$f_d = 9.4636;\ f_d^{early\ exercise} = \mathbf{max}\left(K - S_d, 0\right) = \mathbf{max}\left(52 - 40, 0\right) = 12$$

<u>Early exercise at A</u>

On substituting the revised value of the option at node C as 12 instead of 9.4636, we find that the theoretical value of the option at the node A works out to 5.0894 as below:

$$f_0 = 0.9512\left(0.6282 \times 1.4147 + 0.3718 \times \overbrace{9.4636}^{12.0000}\right) = \overbrace{4.1923}^{5.0894}$$

Now, we examine the payoff if the investor decides to early exercise the option at A. The stock price at A is at 50 while the put's exercise price is 52. Hence, the payoff on early exercise at A will be max(K-S_A,0) =2. Hence, the investor will get a payoff of 2 if he early exercises the put at A. However, if he does not early exercise the put, the theoretical value of the option is 5.0894. Hence, he is better off carrying the option unexercised, since then it commands a theoretical value of 5.0894 compared to an early exercise payoff at A of only 2.

$$f_0^{early\ exercise} = \mathbf{max}\left(K - S_0, 0\right) = \mathbf{max}\left(52 - 50, 0\right) = 2$$

Important to note that at the point A, the value of the American option turns out to be 5.0894 instead of the European option's value of 4.1923. This is because the early exercise property enables the investor to earn a higher payoff at the node C by early exercising compared to carrying it forward unexercised.