Financial Derivatives & Risk Management
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Lecture 44 - Option Pricing: Binomial Model, Risk Neutral Valuation

Option Pricing Models

There are two fundamental models of option pricing viz.

(i) Binomial Model
(ii) Black Scholes Model

The binomial model is the discrete time, discrete variable model while the Black Scholes model is the continuous time, continuous variable version.

Philosophy of option pricing

The philosophy of option pricing is based on the premise that since the source of randomness in both, the underlying asset and the derivative is the same, it should be possible by taking appropriate positions in them to create a portfolio wherein the randomness in one annuls the randomness in the other. In other words,

We can create a risk free asset through a combination of a long/short position in a derivative and an appropriate position in the underlying asset i.e. that it is possible to annul the volatility in the price of the derivative by a counter position in the underlying asset and vice versa.

This is possible because the source of randomness in both price processes is modelled identically.

In other words, we can write, in a rough sense:

Derivative+Δx*Underlying Asset = Riskfree asset or
Derivative = Δx*Underlying Asset+Riskfree asset

Thus, the above is equivalent to contending that we can construct a derivative synthetically by taking appropriate positions in the underlying asset and riskfree bonds.

So, to reiterate, because the randomness embedded in both, the underlying asset and the derivative originates from the same source or is modeled by the same Brownian Motion or the same random walk, we assume that a portfolio can be constructed comprising of positions in the two assets, the derivative and the underlying, such that the randomness is annulled.

The Binomial model

We start with a single period binomial model. For this purpose, we consider a stock priced at S_0 at t=0. Since this is a single period model, let at the end of one period of time of length T i.e. at t=T, the stock price takes one of two possible values viz. $S_u=S_0u$ or $S_d=S_0d$. Further, since we are considering binomial model, the stock can take no other value except S_u or S_d at t=T. Also, there will be no change in stock price in the interval (0,T) since this is a single period model. Only at the end of this single period i.e. at t=T, the stock price will jump from S_0 to either S_u or S_d.

We, now, construct a riskfree asset by formulating a hedge as follows:

We construct a portfolio V at t=0 consisting of:

(i) one unit of derivative, costing f_0 at t=0 and
(ii) Δ units of stock costing S_0 per unit.

The value of this portfolio at t=0 is $V_0= f_0 + \Delta S_0$.

Now, the value of this portfolio V at t=T will be $V_T^u = f_T^u+\Delta S_u$ if the stock price goes up; and $V_T^d = f_T^d+\Delta S_d$ if the stock price goes down where f_T^u and f_T^d are the payoffs from the derivative at t=T if the stock price goes up and down respectively at t=T.

Now, if the portfolio V is to be riskless, its value at t=T must be independent of the up and down movement of the stock price.

This gives: $V_T^u = f_T^u+\Delta S_u=f_T^d+\Delta S_d =V_T^d$ so that $\Delta= - (f_T^u - f_T^d)/(S_u-S_d)$.

The negative sign indicates that the position in the underlying will be opposite to that of the derivative.

Now, since portfolio V is riskless, it will generate the riskfree rate of return so that $V_T=V_0\exp(rT)$.

$$\textit{Thus, we have } V_0 = f_0 + \Delta S_0;\ V_T = V_0 e^{rT}$$

$$V_T = f_T^u + \Delta S_u = f_T^d + \Delta S_d = V_0 e^{rT} \text{ so that } \Delta = -\frac{\left(f_T^u - f_T^d\right)}{S_u - S_d} \text{ giving}$$

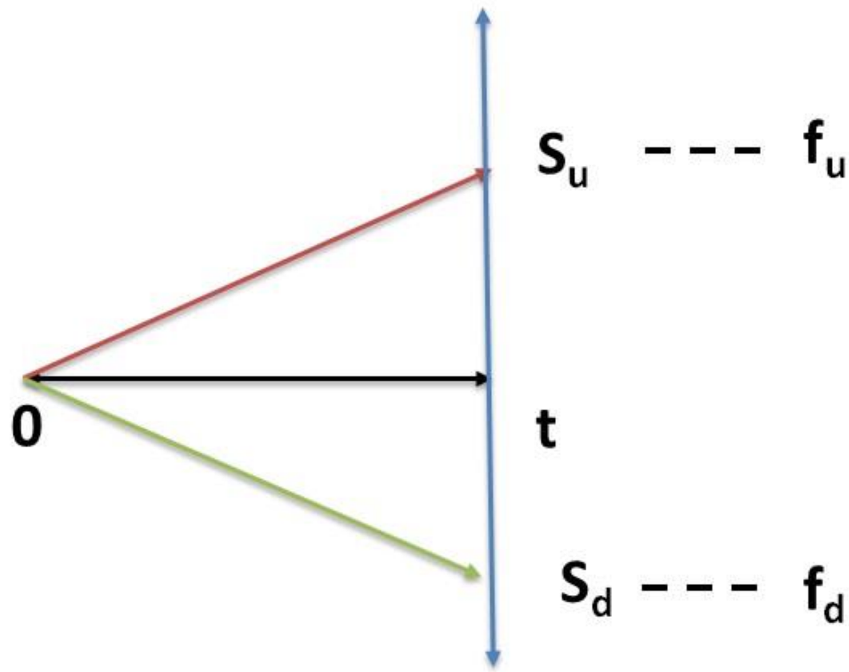
$$f_0 = V_0 - \Delta S_0 = V_T e^{-rT} - \Delta S_0 = \left(f_T^u + \Delta S_u\right)e^{-rT} - \Delta S_0$$

$$= e^{-rT}\,\frac{f_T^d S_u - f_T^u S_d}{S_u - S_d} + \frac{\left(f_T^u - f_T^d\right)}{S_u - S_d} S_0 = e^{-rT}\left[\frac{f_T^d S_u - f_T^u S_d}{S_u - S_d} + \frac{\left(f_T^u - f_T^d\right)}{S_u - S_d} S_0 e^{rT}\right]$$

$$= e^{-rT}\left[f_T^d\left(\frac{S_u - S_0 e^{rT}}{S_u - S_d}\right) + f_T^u\left(\frac{S_0 e^{rT} - S_d}{S_u - S_d}\right)\right] = e^{-rT}\left[q_d f_T^d + q_u f_T^u\right] = e^{-rT} E_Q\left[f\left(S_T\right)\right]$$

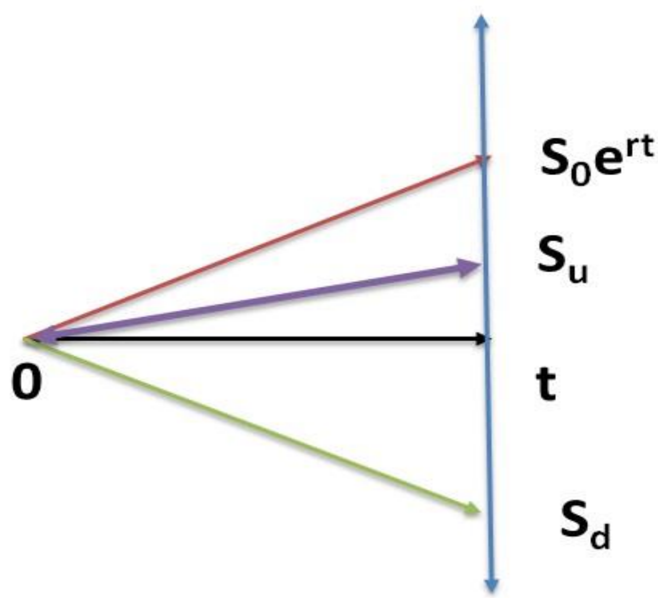
$$\textit{where } q_d = \left(\frac{S_u - S_0 e^{rt}}{S_u - S_d}\right) = \left(\frac{u - e^{rt}}{u - d}\right) \textit{ and } q_u = \left(\frac{S_0 e^{rt} - S_d}{S_u - S_d}\right) = \left(\frac{e^{rt} - d}{u - d}\right)$$

It may be noted that the payoffs f_T^u & f_T^d from the derivative may be positive, zero or negative.



Synthetic probabilities

From above $q_d + q_u = \left(\frac{S_u - S_0 e^{rt}}{S_u - S_d}\right) + \left(\frac{S_0 e^{rt} - S_d}{S_u - S_d}\right) = 1$. Also $0 \le q_d, q_u \le 1$. Why? Let, if possible, $q_d = \left(\frac{S_u - S_0 e^{rt}}{S_u - S_d}\right) < 0$. $\text{Sin}\,ce\ S_u - S_d > 0,\ it\ \ follows\ that\ S_d < S_u < S_0 e^{rT}$.



Now, what is S_u? S_u is the maximum value that stock can take at t=T. In other words, this corresponds to the maximum return that can be generated on the stock. But stock epitomizes the risky asset. Thus, S_u is the price of the risky asset, corresponding to its maximum return i.e. if the stock takes the value S_u at t=T, it yields the maximum return on a risky asset. So if $S_0e^{rT} > S_u$, it follows that the riskfree return is greater than the maximum possible return obtainable on the risky asset. But since the stock is risky, there is a definite probability that it can generate a lower return than that corresponding to S_u. In other words, if an investor invests in the stock, he can get a maximum return corresponding to S_u or he can get a lower return but certainly he cannot get a higher return than that provided by S_u. But if he invests in the riskfree asset, he shall CERTAINLY get a return r which is higher than that corresponding to S_u and hence, higher than any possible return that the stock investment can yield.

Can this happen? The stock is risky and it gives you a lower return than the risk-free asset. Would any rational investor make investment in the stock? No, everybody would go for the risk-free asset. Because risk-free asset is not only giving a higher return, it is also giving that higher return with certainty. And because it is giving higher return with certainty, it is certainly superior to the stock which is giving you a relatively lower return even in its best case scenario.

Thus, this situation cannot sustain in the market. The risk-free asset will increase in demand while the demand for the stock will fall. The prices and, therefore, the returns will start realigning themselves so that $S_d < S_0e^{rT} < S_u$. Similarly, we can prove the other bounds. Thus,

(i) $q_u+q_d=1$;
(ii) $0<q_u, q_d<1$.

We can, therefore, interpret q_u, q_d as some probabilities. we call them synthetic or q-probabilities. Please note that they are purely mathematical constructs and are, in no way, related to real world probabilities. They are simply certain mathematical quantities that seem to satisfy the laws of probability. They are mathematical constructs but they satisfy the axioms of probability. So we call them probabilities but we also call them synthetic probabilities. We shall explore more about them in the following.

<u>Interpretation of synthetic probabilities as risk neutral probabilities</u>

Since we agree to call q_u and q_d as probabilities, we can also define an expectation with respect to these probabilities. Let us call it E_Q. Let us, now, work out the expectation of the stock price with respect to these q-probabilities. We have,

$$E_Q\left(S_T\right) = S_d\left(\frac{S_u - S_0e^{rT}}{S_u - S_d}\right) + S_u\left(\frac{S_0e^{rT} - S_d}{S_u - S_d}\right) = S_0e^{rT}$$

Thus, these synthetic probabilities are such that the expectation for the stock price under these q-probabilities implies a risk-free return. The expectation of the stock price worked out with respect to these synthetic probabilities is such that it gives us a risk-free return.

There are two important implications of this result viz.

(i) Now, stock price is a risky asset. Thus, these probabilities are such that the expected return calculated using them results in a risk-free return. Since this risk-free return is uniform in the market i.e. has the same value for all investors, it follows that all investors, irrespective of the risk profile of the investment, are accepting the same return.

Thus, all the investors do not care at all about risk i.e. each investor is completely risk neutral or indifferent to risk. It is only then that the same return will be acceptable to them even on risky assets.

If all the investors are accepting the same return even on risky assets, it can mean only that they are indifferent to risk. Stock price is a risky asset and still investors are accepting a risk-free return. That clearly means that they are not concerned about the risk. Risk does not constitute a decision criterion for the investors. They take decisions entirely on the basis of expected returns without considering the riskiness of the returns profile of the investment. If an asset gives a higher return, it commands a higher demand even if the "higher" return is highly risky i.e. has significant variability.

In other words, putting it more precisely, investors are risk neutral. Investors are not influenced by risk.

(ii) Why risk-free return? The first thing is that all of the investors are accepting the same return. This implies that the investors are not concerned with the risk of the asset. The second thing is that all investors are accepting the risk-free return. Why it has to be the risk-free return?

The fact that in a risk neutral world, only riskfree return is possible and no higher return follows from arbitrage considerations. If any asset X, other than the riskfree asset F, yields a higher expected return even if it is riskier, all investors will invest in X because, by assumption, they are indifferent to risk whence return is the only consideration for investment. Thus, the demand for X increases while that for F falls. The prices also realign themselves and as a result of which, both the assets will generate the same return.

The return of asset X will be pulled down to the return of F and therefore at the threshold level all investors will receive the risk-free return. Any risky asset will tend to provide a higher expected return than the risk-free return, and as soon as it provides the higher return then the risk-free return, that differential will be eaten away by arbitrage and the expected return will go down to the risk-free level. And all investors will therefore receive only the risk-free return. Because the risk-free asset has zero risk it is supposed to provide the lowest return, this risk-free return will constitute the threshold return for all investors.

We can, in view of the above, interpret these synthetic probabilities, that are associated with risk neutrality as risk neutral probabilities. To reiterate, they are not real-life probabilities. They do not emerge out of actual experimentation and observation. They are simply mathematical constructs.

The bottom-line of the above analysis is that the q-probabilities reflect the probabilities of stock price movements in a risk neutral world i.e. in a world where risk has no significance to each and every investor and return is the only investment criterion. Hence, these probabilities are also called risk neutral probabilities.

Let us call it C, price of the derivative is the present value of the expectation of the payoff from the derivative on the date of maturity; that expectation being calculated with respect to risk neutral probabilities.

Thus, the price of a derivative is the present value of the expected payoff from the derivative on maturity, the expectation being calculated with reference to risk neutral probabilities.

$$c = e^{-rT} E_Q\left[f\left(S_T \right) \right]$$

Why real-life probabilities not relevant

Actually, the derivative's payoff $f(S_T)$ is a deterministic function of the future stock price. However, the future stock price S_T is a stochastic variable. f is deterministic but S_t is stochastic.

Now, the expected stock price at T depends on the probability of upswing/ downswing at T. Hence, the real life probabilities of stock price upswing and downswing are already captured in the stock price. The derivative price being a deterministic function of S_T retains the probabilistic structure of S_T.

Alternative explanation

Our model assumes the practicability of construction of a completely riskfree asset by a combination of the derivative and the underlying. We then price the derivative by pricing this riskfree portfolio (that gives riskfree return) and the stock.

Now, the real world probabilities of upswing/downswing are already encoded in the stock price. The price that prevails as on a particular date encapsulates everything about the perceived future behaviour of the stock and hence, the real world probabilities. Hence, the real world probabilities do not come into play in derivative's pricing.

Equivalently, we constructed a portfolio consisting of the derivative and the stock in such a way that the fluctuations of one annulled those of the other and we created an asset with no fluctuations. We argued that this was equivalent to the riskfree asset and would yield the riskfree return, by no arbitrage. Knowing the stock's valuation and that of the riskfree asset gave us the value of the derivative.

Alternatively, we assume that a derivative can be synthesized by taking appropriate positions in the stock (underlying asset) and the riskfree asset. We, then, price the derivative by pricing this combination i.e. the combination of the stock and the riskfree asset.

The fundamental issue is that the derivative is being priced indirectly by synthesizing an equivalent payoff using the stock and riskfree asset and, then, invoking no arbitrage.

We have not valued the derivative in isolation, we have valued a portfolio consisting of the stock and the risk-free asset as equivalent to the derivative. And then we said that because, the two were giving identical payoffs, they will cost the same.

Therefore, we, because we are valuing the derivative, not directly as the derivative itself but as a no arbitrage combination of the risk-free asset and the stock, and the stock contains the information on the real world probabilities, we do not need to consider real world probabilities in the context of the derivative.