Financial Derivatives and Risk Management
Professor J.P. Singh
Indian Institute of Technology Roorkee
Lecture 43
Lognormal Distribution

Calculus of functions of stochastic variables

Just as deterministic curves can be considered as comprising of infinitesimal increments of straight lines, stochastic trajectories may be assumed to be formed by infinitesimal increments of Brownian motion.

In other words, deterministic curves are an assortment, a combination of infinitesimal straight lines. Similarly, stochastic curves or curves representing random processes or stochastic processes can be considered as increments of infinitesimal Brownian motion.

The following correspondence is very interesting:

$$E(dx) = dx;\ E(dx)^2 \approx 0;\ Var\ (dx) = 0;\ Var\ (dx)^2 = 0$$
$$E(dW_t) = 0;\ E(dW_t)^2 \approx dt;\ Var\ (dW_t) = dt;\ Var\ (dW_t)^2 \approx 0$$

The above is a comparison between the deterministic scenario and the stochastic scenario. If dx is a deterministic increment, then obviously because it has no randomness, the E(dx)=dx and Var(dx)=0. Further, in Newtonian calculus, we ignore higher powers of dx so the $E(dx^2)=0$ and we truncate the Taylor series at the first order term. Further, $Var(dx^2)=E(dx^4)-[E(dx^2)]^2=0$. The results on variance also follow intuitively as dx has no randomness, it is a deterministic increment.

In the case of BM, however, we have a different situation. We have, E[dW(t)]=0, by definition but $E[dW(t)]^2=dt$. It is not zero, it does not vanish. Rather it is of first order in dt. Thus, while $E[(dx)^2]$ is ignored, it is assumed to be small enough, $E[dW(t)]^2=dt$ cannot be ignored because it is of first order in dt, Besides, $Var[dW(t)]^2$ is of order $(dt)^2$ showing that the variance of $[dW(t)]^2$ can, indeed, be ignored. Thus, $[dW(t)]^2$ has the mean value dt and is approximately non-stochastic, non-random. It follows that $[dW(t)]^2$ has the value dt throughout, whence being of first order it must be retained in taking limits.

Thus, in the case of dW(t) the increment of the Brownian motion, howsoever small it may be, $dW(t)^2$ cannot be ignored because it is of the first order in dt it is approximately non-stochastic and therefore it has to be retained in the Taylor expansion and while taking limits in doing differentiation.

Analysis of stock price model

The infinitesimal stock price model was assumed as:

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW \ \ or\ \frac{1}{dt}\left(\frac{dS}{S}\right) = \mu + \sigma\eta(t)$$

where $\eta(t)$ is Gaussian delta correlated white noise.

The corresponding finite model shows the stock price to be log normally distributed as:

$$p\left(S,t\middle|S',t'\right)=\frac{1}{\sqrt{2\pi\left(\sigma S\right)^{2}\left(t-t'\right)}}\times\exp\left\{-\frac{\left[\ln\left(\frac{S}{S'}\right)-\left(\mu-\frac{\sigma^{2}}{2}\right)\left(t-t'\right)\right]^{2}}{2\sigma^{2}\left(t-t'\right)}\right\}$$ i.e.

$$\ln S_T \xrightarrow{distribution} N\left[\ln S_0+\left(\mu-\frac{1}{2}\sigma^2\right)T,\sigma^2 T\right]$$

Now the important to highlight that in the lognormal model, the mean and variance are independent of the stock price whereas in the differential model the they are both functions of the instantaneous stock price. This is the reason that the differential model cannot be extended for long time periods.

It is interesting to compare the above two models. In the following example, I have tried to analyse the results by applying the two models to the same set of data in two situations.

The data that I have used is μ=15%,, σ=30%, S_0=1000. In the first case, I consider a time frame of 1 week= 0.0192 years and work out the probability of the stock price rising by 54 at the end of this period using both the models. With the differential model, I get Z=1.2286 whereas with the lognormal model I get Z= 1.21. The difference, though clearly discernible, is relatively small.

ANALYSIS OF STOCK PRICE MODELS	
BASIC DATA	
INSTANTANEOUS RETURN	0.15
STD DEVIATION PER ANNUM (Vol)	0.3
CURRENT STOCK PRICE	1000
EXPECTED STOCK PRICE	1054
TIME FRAME	0.0192308
SHORT TERM MODEL	
MEAN	1002.8846
SD	41.602515
Z VALUE	1.2286609
LONG TERM MODEL	
LN (S)	6.9077553
MEAN OF LN(S)	6.9097745
SD OF LN(S)	0.0416025
LN(S(T))	6.9603477
Z VALUE	1.2156289
	0.013032

Now, I rework the same problem but with a time period of one year i.e. I work out the probability of the stock price exceeding 1,054 at the end of one year from today given its today's price as 1,000 and μ and σ as 15% and 30% respectively. With the differential model, I return the Z value as -0.32 while the lognormal model gives Z=-0.14, thus showing significant divergence at this time-scale.

The two results clearly establish that the differential model is valid for only very small periods of time.

BASIC DATA						
INSTANTANEOUS RETURN						0.15
STD DEVIATION PER ANNUM (Vol)						0.3
CURRENT STOCK PRICE						1000
EXPECTED STOCK PRICE						1054
TIME FRAME						1
SHORT TERM MODEL						
MEAN						1150
SD						300
Z VALUE						-0.32
LONG TERM MODEL						
LN (S)						6.90776
MEAN OF LN(S)						7.01276
SD OF LN(S)						0.3
LN(S(T))						6.96035
Z VALUE						-0.17469
						-0.14531

The basic thing is, the correct model of the two is unquestionably the lognormal model. The differential model, however, offers a good approximation for small time-scales. Therefore, I emphasize that while working out probabilities or confidence intervals if the period involved is substantial, say exceeding 1 week, it is definitely better to use the lognormal model.

Now we investigate, why this difference arises due to the use of the two models. Let us look at an example: Let the stock price at t=0 i.e. S_0=1,000 and let the price at the end of one-time period (say 1 year) at t=1 i.e. S_1=1,000. Naturally, since there is no net change in the stock price over the period, the net return is 0.

Now, suppose that we split this time period of 1 year into two parts of 6 months each i.e. 0 to 0.50 year and then 0.50 year to 1 year. Let the stock price at t=0.50 year be 1,100 i.e. $S_{0.50}$=1,100 and let it drop down to 1,000 at t=1 year so that the net return is again 0. But the percentage change in price over the first half year and the second half year are respectively 10% and -9.09% with a total change of 0.91%, clearly incompatible with the zero net change in price and zero return.

Now, I do the same exercise with the stock price instead of going up to 1100, but going down to 900 at t=0.50 year and then recouping to 1,000 at t=1 year. In this case, the percentage change in price over the two half year periods is respectively -10% and 11.11% with a total change over the year as 1.11%. This is, again, incompatible with the zero net price change and net return.

Thus, in both cases, if the stock price goes up or down at a point in between the period for which the return is being calculated, then the return over the entire period calculated by aggregating the percentage change in prices of the two periods is more than the return calculated by using the net change in price over the entire period.

The above can be generalized so that if there occur fluctuations during a period, then the return computed over the period by aggregating the percentage price changes over the periods at which the fluctuations occur works out to be more than the return computed by using the net absolute price change over the period. In fact, this explains the need for the correction of ½σ^2.

YEARS			0	0.5	1		
STOCK PRICE			1000	1100	1000		
RETURN				0.2	-0.182	0.00909	
STOCK PRICE			1000	900	1000		
RETURN				-0.2	0.2222	0.01111	0.00202
STOCK PRICE			1000	1075	1150		
RETURN				0.15	0.1395	0.14477	
STOCK PRICE			1000	925	1150		
RETURN				-0.15	0.4865	0.16824	0.02348
			0	0.33	0.66	1	
STOCK PRICE			1000	1050	1100	1150	
RETURN				0.15152	0.1443	0.13774	0.14452
							0.00025

There is another interesting observation. In both the cases discussed above, the stock price was shown to make an up (down) jump followed by a reversal down (up) jump i.e. the two jumps were in opposite directions.

Now, suppose the stock price makes the two jumps in the same direction i.e. from 1,000 to 1,075 at the mid-point (t=6 months) and then to 1,150 at the end of one year. In this case, the percentage changes in price work out to 7.5% for the first half year and (-)6.98% for the second half year yielding an aggregate percentage price change of 14.48% for the year. This is against the net return of 15% over the year clearly showing that in this case i.e. when both price movements are unidirectional, the actual return exceeds the averaged return over the two periods. The same phenomenon happens when both the jumps are down. A generalized analysis is as follows:

$$R_{AC} = \frac{dS_{AC}}{S_A} = \frac{dS_{AB} + dS_{BC}}{S_A}$$

$$R_{AB} + R_{BC} = \frac{dS_{AB}}{S_A} + \frac{dS_{BC}}{S_B} = \frac{dS_{AB}}{S_A} + \frac{dS_{BC}}{S_A + dS_{AB}}$$

$$= \frac{dS_{AB}\left(S_A + dS_{AB}\right) + S_A dS_{BC}}{S_A\left(S_A + dS_{AB}\right)} = \frac{S_A dS_{AB} + \left(dS_{AB}\right)^2 + S_A dS_{BC}}{S_A\left(S_A + dS_{AB}\right)}$$

$$= \frac{S_A dS_{AB} + S_A dS_{BC}}{S_A\left(S_A + dS_{AB}\right)} + \frac{\left(dS_{AB}\right)^2}{S_A\left(S_A + dS_{AB}\right)}$$

$$= \frac{S_A dS_{AB} + \left(dS_{AB}\right)^2 + S_A dS_{BC}}{S_A\left(S_A + dS_{AB}\right)} > \frac{dS_{AB} + dS_{BC}}{S_A} \ if$$

$$R_{AB} + R_{BC} > R_{AC} \ if \ dS_{AB} dS_{BC} < 0$$

This is an explanation of what we have just discussed. We have 3 points here A, C, and an intermediate B. The return over period AC is given by the net percentage change in price over AC i.e.R_{AC}= dS_{AC}/S_A. But dS_{AC} = dS_{AB}+ dS_{BC} so that R_{AC}=(dS_{AB}+ dS_{BC})/S_A.

But, R_{AB}= dS_{AC}/S_A and R_{BC}= dS_{BC}/S_B so that R_{AB}+R_{BC}= (dS_{AC}/S_A)+(dS_{BC}/S_B).

Importantly, while working out the percentage change in price over BC we need to take the denominator as S_B instead of S_A. This is important. On simplification, we see that:

$$R_{AB} + R_{BC} > R_{AC} \ if \ dS_{AB} dS_{BC} < 0$$

Thus, if the sum of the percentage changes in price over AB & BC is to be greater than the net percentage change in price over AC, then we must have that $dS_{AB}dS_{BC}$ <0 i.e. that there should be up and down movement of the stock or vice versa, the fluctuation in price should be bidirectional. The signs of dS_{AB} and dS_{BC} or the sign of the change in the price over AB and over BC must be in opposite direction. So that is what happened in the illustrations.

<u>Lognormal Distribution, its PDF, mean & variance</u>

$$\textit{Let } X \textit{ be } \mathbf{log}\textit{normally distributed with pdf } \rho(x)\mathbf{.}$$

$$\textit{Then } Y = \mathbf{ln}\, X \textit{ is normally distributed}\mathbf{.}$$

$$\textit{Let } y \xrightarrow{distribution} N\left(\lambda, \theta^2\right)$$

$$p(y) = \frac{1}{\sqrt{2\pi\ \theta^2}} \mathbf{exp}\left[-\frac{\left(y - \lambda\right)^2}{2\theta^2}\right]$$

$$P\left(x < X < x + dx\right) = \rho(x)\,dx = P\left(\mathbf{ln}\, x < \mathbf{ln}\, X < \mathbf{ln}\left(x + dx\right)\right)$$

$$= P\left(y < Y < y + dy\right) = p(y)\,dy = p\left(\mathbf{ln}\, x\right)\left[\mathbf{ln}\left(x + dx\right) - \mathbf{ln}\, x\right]$$

$$= p\left(\mathbf{ln}\, x\right)\left[\mathbf{ln}\left(1 + \frac{dx}{x}\right)\right] = p\left(\mathbf{ln}\, x\right)\frac{dx}{x} = \frac{dx}{x\sqrt{2\pi\ \theta^2}} \mathbf{exp}\left[-\frac{\left(\mathbf{ln}\, x - \lambda\right)^2}{2\theta^2}\right]$$

$$E(X)=\int_0^\infty xf(x)dx=\int_0^\infty x\frac{1}{x}\frac{1}{\sqrt{2\pi}\,\theta}\exp\left[-\frac{(\ln x-\lambda)^2}{2\theta^2}\right]dx$$

$$=\frac{1}{\sqrt{2\pi}\,\theta}\int_{-\infty}^{\infty}\exp\left[-\frac{(u-\lambda)^2}{2\theta^2}\right]e^u du \qquad \left(Put\ u=\ln x\right)$$

$$=\frac{1}{\sqrt{2\pi}\theta}\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{(u-\lambda)^2-2u\theta^2}{2\theta^2}\right]\right\}du=\frac{1}{\sqrt{2\pi}\theta}\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{(u-\lambda-\theta^2)^2-2\lambda\theta^2-\theta^4}{2\theta^2}\right]\right\}du$$

$$=\exp\left(\frac{2\lambda\theta^2+\theta^4}{2\theta^2}\right)\frac{1}{\sqrt{2\pi}\theta}\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{(u-\lambda-\theta^2)^2}{2\theta^2}\right]\right\}du=\exp\left(\lambda+\frac{\theta^2}{2}\right).$$

The last step follows because $\frac{1}{\sqrt{2\pi}\theta}\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{(u-\lambda-\theta^2)^2}{2\theta^2}\right]\right\}du=1$

as $\frac{1}{\sqrt{2\pi}\theta}\exp\left\{-\left[\frac{(u-\lambda-\theta^2)^2}{2\theta^2}\right]\right\}du$ *is the probability density of a normal variable with mean* $\lambda+\theta^2$ *and standard deviation* θ.

$$E(X^2)=\int_0^\infty x^2 f(x)dx=\int_0^\infty x\frac{1}{\sqrt{2\pi}\theta}\exp\left[-\frac{(\ln x-\lambda)^2}{2\theta^2}\right]dx$$

$$=\frac{1}{\sqrt{2\pi}\theta}\int_{-\infty}^{\infty}e^u\exp\left[-\frac{(u-\lambda)^2}{2\theta^2}\right]e^u du=\frac{1}{\sqrt{2\pi}\theta}\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{(u-\lambda)^2-4u\theta^2}{2\theta^2}\right]\right\}du$$

$$=\frac{1}{\sqrt{2\pi}\theta}\int_{-\infty}^{\infty}\exp\left\{-\left[\frac{(u-\lambda-2\theta^2)^2}{2\theta^2}-\frac{4\lambda\theta^2+4\theta^4}{2\theta^2}\right]\right\}du$$

$$=\exp\left(\frac{4\lambda\theta^2+4\theta^4}{2\theta^2}\right)\frac{1}{\sqrt{2\pi}\theta}\int_{-\infty}^{\infty}\exp\left[-\frac{(u-\lambda-2\theta^2)^2}{2\theta^2}\right]du=\exp(2\lambda+2\theta^2)$$

$$V(X)=E(X^2)-\left(E(X)\right)^2=\exp(2\lambda+\theta^2)\left[\exp\left(\theta^2\right)-1\right]$$

In our stock price model, we have $\ln S_T \xrightarrow{distribution} N\left[\ln S_0+\left(\mu-\frac{1}{2}\sigma^2\right)T,\sigma^2T\right]$ so that:

$$\lambda=\ln S_0+\left(\mu-\frac{1}{2}\sigma^2\right)t;\ \theta=\sigma\sqrt{t},\ and\ so$$

$$E\left(S_t\right)=S_0\exp\left(\mu t\right)\ and\ Var\left(S_t\right)=S_0^2\exp\left(2\mu t\right)\left[\exp\left(\sigma^2 t\right)-1\right]$$

<u>Example 1</u>

A stochastic process is modelled as unscaled Brownian motion without drift i.e. standard Brownian motion in one dimension (μ=0, σ=1). What is the probability of the process being more than 1 unit away from its initial value, after 25 units of time?

<u>Solution</u>

Let X represent the displacement of the particle form the origin.
Then, $\mu=E(X)=0$, SD (s)=$\sigma\sqrt{T}=5$
We need to find $1-P(-1<X<+1)=1-P(-0.2<Z<+0.2)$
From the normal tables: $1-P(-0.2<z<+0.2)=0.8414$.

Example 2

A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.5 per quarter and a variance rate of 4.0 per quarter. What is the minimum initial cash position of the company (in millions) such that it has a less than 5% chance of a negative cash position by the end of one year?

Solution

Suppose that the company's initial cash position is x. Then, the closing cash position, X, is normally distributed with a mean of $x+4*0.50=x+2$ and a variance of $4*4=16$.

We are given that the probability of the closing cash balance, X, being negative i.e.$X<0$ should not exceed 0.05. Thus, we are given that:

$P(X<0)=0.05$ where X is $N(x+2,16)$. The standard normal variate corresponding to X=0 is $[0-(x+2)]/4=-(x+2)/4$. We are thus given that: $P(Z<-(x+2)/4)=0.05$.

From the normal distribution table, we see that 5% of area to the left of z i.e. $P(Z<z)=0.05$ corresponds to a z value of -1.645. Hence. We must have:

$-(x+2)/4=-1.645$ or $x=4.58$ million.