Financial Derivatives and Risk Management
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Lecture 42 - Stock Price Distributions; Fokker Planck Equation & Solution

Assumptions underlying stock [price distributions
Infinitesimal time model

(i) Stock prices are assumed to follow a Markov process. The Markov property of stock prices is consistent with the weak form of market efficiency i.e. that the current market price encapsulates its entire past history and, therefore, the current price is the only relevant information for future price prediction. Whatever is today's stock price encapsulates everything or incorporates all information about the prior behaviour of the price.

(ii) The expected instantaneous percentage return i.e. the percentage change in price (dS/S) over an infinitesimal time interval dt required by investors from a stock is independent of the stock's price. If investors require a 15% per annum expected return when the stock price is 10, then, they will also require a 15% per annum expected return when it is 50. Thus, we can write:

$$E(r) = E\left[\frac{1}{dt}\left(\frac{dS}{S}\right)\right] = \mu \ or \ E\left(\frac{dS}{S}\right) = \mu dt$$

$$\frac{dS}{S} = \mu dt + X \ \ (where\ X\ is\ a\ random\ \text{var}iable\ with\ zero\ mean)$$

(iii) The variance of instantaneous percentage return over infinitesimal time dt is constant and independent of the stock price. An investor is just as uncertain of the instantaneous percentage return when the stock price is 50 as when it is 10.

$$Var(r) = Var\left(\frac{1}{dt}\frac{dS}{S}\right) = \sigma^2 \ \ so\ that\ SD\left(\frac{dS}{S}\right) = \sigma\sqrt{dt}$$

Combing (ii) & (iii), we can write:

$$\frac{dS}{S} = \mu dt + \sigma dW = \mu dt + \sigma z\sqrt{dt}$$

$$Clearly\ E\left(\frac{dS}{S}\right) = E(\mu dt + \sigma dW) = E\left(\mu dt + \sigma z\sqrt{dt}\right) = \mu dt\ \ \text{sin}ce\ E(z) = 0$$

$$Var\left(\frac{dS}{S}\right) = Var(\mu dt + \sigma dW) = Var\left(\sigma z\sqrt{dt}\right) = \sigma^2 dt Var(z) = \sigma^2 dt\ \ \text{sin}ce\ Var(z) = 1$$

The model $\frac{dS}{S} = \mu dt + \sigma dW = \mu dt + \sigma z\sqrt{dt}$ is the model of stock prices that we usually employ when desirous of modelling these prices over small time scales. As per this model, (dS/S) i.e. percentage change in stock price is normally distributed with a mean of μdt and a variance of σ^2dt i.e. (dS/S is N(μdt, σ^2dt). Given S at time t i.e. S_t, since $dS=S_{t+dt} - S_t$, it follows that S_{t+dt} is normally distributed with mean $S_t+\mu S_t dt$ and a variance of $\sigma^2 S_t^2 dt$ i.e that S_{t+dt} is $N(S_t+\mu S_t dt, \sigma^2 S_t^2 dt)$.

However, it is strongly emphasized that this model holds for infinitesimal (small) time scales.

Finite time model

We use the Ito's lemma to arrive at a finite time version of the aforesaid infinitesimal model of stock prices. We assume $G(S,t)=\ln S$ where $dS=\mu Sdt+\sigma SdW$. Using Ito's Lemma find the drift and diffusion terms and the distribution of G(S,t).

Ito's Lemma states that if G(x,t) is a twice differentiable function of x where x is given by dx=adt+bdW, then

$$dG=\left(a.\frac{\partial G}{\partial x}+\frac{\partial G}{\partial t}+\frac{1}{2}b^2.\frac{\partial^2 G}{\partial x^2}\right)dt+b.\frac{\partial G}{\partial x}dW_t$$

In our case, $G(S,t)=\ln S$, $\frac{\partial G}{\partial S}=\frac{\partial \ln S}{\partial S}=\frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2}=\frac{\partial}{\partial S}\left(\frac{1}{S}\right)=-\frac{1}{S^2}$, $\frac{\partial G}{\partial t}=\frac{\partial \ln S}{\partial t}=0$, $a=\mu S$, $b=\sigma S$. Putting the values, we get:

$$dG=d(\ln S)=\left(\mu S\frac{1}{S}-\frac{1}{2}\sigma^2S^2\frac{1}{S^2}\right)dt+\sigma S\frac{1}{S}dW_t=\left(\mu-\frac{1}{2}\sigma^2\right)dt+\sigma dW_t$$

This shows that the drift term of $G(S,t)=\ln S$ is $\left(\mu-\frac{1}{2}\sigma^2\right)$ and the diffusion term is σ. Further, $d(\ln S)$ is normally distribution with a mean of $\left(\mu-\frac{1}{2}\sigma^2\right)dt$ and a variance of $\sigma^2 dt$.

Equivalently $\ln S_T$ is $N\left[\ln S_0+\left(\mu-\frac{1}{2}\sigma^2\right)T,\sigma^2T\right]$.

At this point it is pertinent to highlight that the primary difference between the two models is the appearance of the σ^2 term in the latter model. Tracking its origin backwards, it is seen to arise due to the second derivative term $\frac{\partial^2 G}{\partial S^2}=\frac{\partial}{\partial S}\left(\frac{1}{S}\right)=-\frac{1}{S^2}$ in the total differential of G=ln S. This, as explained earlier is the hallmark of stochastic processes.

Now, the fact that ln S_T is normally distributed, means S_T is log normally distributed. Hence, we arrive at the famous log normal distribution of stock prices.

Distribution of returns

$$\textit{We define the} \log \textit{arithmic return as } r_{\ln}=\frac{1}{T}\ln\frac{S_T}{S_0}.\ \text{Sin}\,ce,$$

$$\ln\ S_T\xrightarrow{distribution}N\left(\ln\ S_0+\left(\mu-\frac{1}{2}\sigma^2\right)T,\sigma^2T\right)$$

$$\textit{it follows that } r_{\ln}=\frac{1}{T}\ln\frac{S_T}{S_0}\xrightarrow{distribution}N\left(\left(\mu-\frac{1}{2}\sigma^2\right),\frac{\sigma^2}{T}\right)$$

So the stock prices are log normally distributed but the logarithmic returns are normally distributed.

Relation between the infinitesimal and finite time model

We define the logarithmic return over a time dt as $r_{\ln} = \frac{1}{T}\ln\frac{S_T}{S_0}$ which gives us $S_T = S_0 \exp\left(r_{\ln}T\right) = S_0\left(1 + r_{\ln}T + \frac{1}{2}r_{\ln}^2 T^2 + \ldots\right)$. On the other hand, the definition of return that we employed in the infinitesimal model was: $r = \frac{1}{dt}\frac{dS}{S} = \frac{1}{dt}\frac{S_{t+dt} - S_t}{S_t}$ *or* $S_{t+dt} = S_t\left(1 + rdt\right)$. Clearly, the former model translates to the latter in the limit that the second order and higher terms in $r_{\ln}T$ are small enough to be insignificant and therefore be ignored i.e. when T is very small. Equivalently, the infinitesimal model is the first order approximation of the log-normal model.

Modelling of assets with continuous yields

In case the stocks or other similar assets (e.g. currencies) are such that they generate continuous yields @ q per unit time compounded continuously, the above models get modified respectively to:

$dS = \left(\mu - q\right)Sdt + \sigma SdW$ (the infinitesimal approximation)

$\ln S_T \xrightarrow{distribution} N\left[\ln S_0 + \left(\mu - q - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right]$ (finite time model)

In other words, all the above analysis will hold with the replacement $\mu \to \mu - q$.

Example 1

Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continuous compounding and has the current price of INR 1,000. Calculate the probability that the stock price will increase by INR 36 or more in one week from now. Assume that "one week" qualifies as an "infinitesimally small" time period so that we can use the usual stock price model for infinitesimally small time periods. Also assume 1 week = 0.0192 year.

Solution

$$\text{In the } \mathbf{inf}\text{ initesimal } \mathbf{mod}\text{el}: dS = \mu Sdt + \sigma SdW \text{ so that}$$
$$dS \xrightarrow{distribution} N\left[\mu S_0 dt, \sigma^2 S_0^2 dt\right]$$

In the given problem S_0=1,000, μ=0.15, σ=0.30, dt =0.0192. Hence,

$$dS \xrightarrow{distribution} N\left[\mu S_0 dt, \sigma^2 S_0^2 dt\right]$$
$$= N\left(0.15 \times 1000 \times 0.0192, 0.30^2 \times 1000^2 \times 0.0192\right)$$
$$= N\left(0.15 \times 1000 \times 0.0192, 0.30^2 \times 1000^2 \times 0.0192\right)$$
$$= N\left(2.88, 1728\right)$$

Thus, dS is normally distributed with a mean of 2.88 and a standard deviation of 41.57.

We need to find out P(dS>36)=P(Z>(36-2.88)/41.57)=P(Z>0.80)=0.2119

Example 2

A stock price follows a lognormal distribution with an expected rate of return μ of 14% and a volatility of 30% p.a. The stock pays dividends at a rate of 2% p.a. (with continuous compounding). The current price of the stock is INR 1,000. Calculate the expected price of the stock after six months.

Solution

The stock price is distributed as follows:

$$\ln S_T \xrightarrow{distribution} N\left[\ln S_0 + \left(\mu - q - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right]$$

Hence, $E\left(\ln S_T\right) = \ln S_0 + \left(\mu - q - \frac{1}{2}\sigma^2\right)T$

Now, by the properties of lognormal distribution (See Appendix) $\ln E\left(X\right) = E\left(\ln X\right) + \frac{1}{2}Var\left(\ln X\right)$ so that $\ln E\left(S_T\right) = E\left(\ln S_T\right) + \frac{1}{2}\sigma^2 T$. Thus,

$\ln\left[E\left(S_T\right)\right] = \ln S_0 + \left(\mu - q\right)T$ or

$$E\left(S_T\right) = \exp\left[\ln S_0 + \left(\mu - q\right)T\right] = \exp\left(\ln S_0\right)\exp\left[\left(\mu - q\right)T\right]$$
$$= S_0 \exp\left[\left(\mu - q\right)T\right] = 1000 \times \exp\left[\left(0.14 - 0.02\right) \times 0.50\right] = 1062$$

Example 3

A stock price follows a lognormal distribution with an expected rate of return μ of 14% and a volatility of 30% p.a. The stock pays dividends at a rate of 2% p.a. (with continuous compounding). The current price of the stock is INR 1,000. Calculate the probability that the stock price will exceed INR 1,100 at the end of six months from now.

Solution

The stock price is distributed as follows:

$\ln S_T \xrightarrow{distribution} N\left[\ln S_0 + \left(\mu - q - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right]$. In the given problem S_0=1,000, μ=0.14, q=0.02, σ=0.30 and T=0.50 so that

$$\ln S_T \xrightarrow{distribution} N\left[\ln 1000 + \left(0.14 - 0.02 - \frac{1}{2}\times 0.30^2\right)\times 0.50, 0.30^2 \times 0.50\right]$$

$$= N\left[6.9078 + 0.0375, 0.045\right] = N\left[6.9453, 0.045\right]$$

We need to find $P(S_T>1,100)=P(\ln S_T>7.003)=P(Z>(7.003-6.9453)/0.045)=P(Z>1.28)$ =0.1003

Example 4

A stock price follows a lognormal distribution with an expected rate of return μ of 14% and a volatility of 30% p.a. The stock pays dividends at a rate of 2% p.a. (with continuous compounding). The current price of the stock is INR 1,000. Calculate the 95% confidence level for the stock returns at the end of six months from now.

Solution

The stock returns are normally distributed as follows:

$r_{\ln} = \frac{1}{T}\ln\left(\frac{S_T}{S_0}\right) \xrightarrow{distribution} N\left[\left(\mu - q - \frac{1}{2}\sigma^2\right), \frac{\sigma^2}{T}\right]$. In the given problem S_0=1,000, μ=0.14, q=0.02, σ=0.30 and T=0.50 so that

$$r_{\ln} = \frac{1}{T}\ln\left(\frac{S_T}{S_0}\right) \xrightarrow{distribution} N\left[\left(0.14 - 0.02 - \frac{1}{2}\times 0.30^2\right), \frac{0.30^2}{0.50}\right] = N\left[(0.075), 0.18\right]$$

The 95% confidence interval for the return will be given by $E\left(r_{\ln}\right) \mp 1.96\sigma_{r_{\ln}} = 0.075 \mp 0.83$

Fokker Planck equation

In the above derivation, Ito's equation has been used to arrive at the lognormal distribution of stock prices. We started with G=ln S. Why? This issue makes the derivation somewhat obscure. We do the same derivative using a more explicit & intuitive approach viz the Fokker Planck equation:

The stock price equation $dS = \mu S dt + \sigma S dW(t)$ where dW(t) is standard Brownian motion increment and E[W(t)]=0, E[W(t),W(t')]=min(t,t') is a stochastic differential equation. It can also be written in the form $\frac{1}{dt}\left(\frac{dS}{S}\right) = \mu + \sigma\eta(t)$ which is a special case of the generalized Langevin equation $dx(t) = f(x)dt + g(x)dW(t)$ or $\frac{dx(t)}{dt} = f(x) + g(x)\eta(t)$. $\eta(t)$ is delta correlated Gaussian noise $E\left[\eta(t), \eta(t')\right] = \delta(t-t')$. In some sense, $\eta(t)$ is the derivative of Brownian motion, although strictly speaking, Brownian motion is not differentiable at any point. Therefore, rigorously speaking, there is no mathematical derivative of Brownian motion but $\eta(t)$ serves something like a derivative for Brownian motion at a conceptual level although technically, it is not correct.

We shall start with a general form of the Langevin equation $dx(t) = f(x)dt + g(x)dW(t)$, obtain the corresponding Fokker Planck equation, then write and solve the Fokker Planck equation for our stock price equation.

We, therefore, start with the Langevin equation:

$$dx(t) = f(x)dt + g(x)dW(t)$$

Discretising the above equation, we obtain,

$$x(t+dt) - x(t) = f(x)dt + g(x)z\sqrt{dt}; \;\; z \; is \; N(0,1)$$

Let G(x(t) be an arbitrary function of x(t), then,

$$\frac{\partial}{\partial t}G(x(t)) = \lim_{dt\to 0}\frac{G(x(t+dt)) - G(x(t))}{dt}$$

But $x(t+dt) = x(t) + f(x)dt + g(x)z\sqrt{dt}$ so that

$$\frac{\partial}{\partial t}G(x(t)) = \lim_{dt\to 0}\frac{G(x(t+dt)) - G(x(t))}{dt}$$
$$= \lim_{dt\to 0}\frac{G\left(x(t) + f(x)dt + g(x)z\sqrt{dt}\right) - G(x(t))}{dt}$$
‘

Taylor expanding $G\left(x(t) + f(x)dt + g(x)z\sqrt{dt}\right)$, we get

$$G\left(x(t) + f(x)dt + g(x)z\sqrt{dt}\right) = G(x(t)) + G'(x)\left[f(x)dt + g(x)z\sqrt{dt}\right] +$$
$$\frac{G''(x)}{2}\left[f(x)dt + g(x)z\sqrt{dt}\right]^2$$

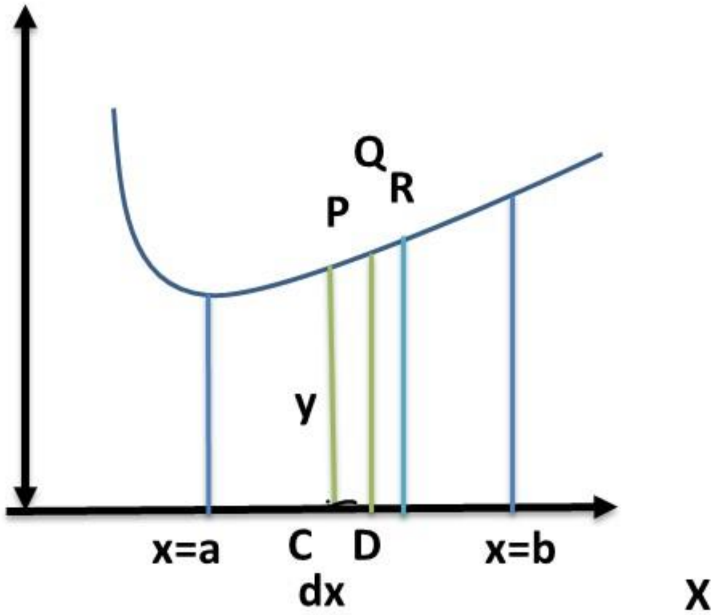
Hence,

$$\frac{\partial}{\partial t}G(x(t)) = \lim_{dt\to 0}\frac{1}{dt}\left\{\begin{matrix} G'(x)\left[f(x)dt + g(x)z\sqrt{dt}\right] + \\ \frac{G''(x)}{2}\left[f(x)dt + g(x)z\sqrt{dt}\right]^2 \end{matrix}\right\}$$
$$= \lim_{dt\to 0}\frac{1}{dt}\left[G'(x)f(x)dt + G'(x)g(x)z\sqrt{dt} + \frac{G''(x)}{2}g(x)^2 z^2 dt + O\left((dt)^{3/2}\right)\right]$$

Now, we make the Ito assumption i.e. that when discretizing the Langevin eq we compute g(x(t)) at the beginning of the time step, i.e. using the value t, and not (t+dt)/2.

To understand the Ito assumption, let us look back at the integral expression for area viz. $A=\int_{x=a}^{x=b} y dx=\lim_{n\to\infty}\sum_{i=1}^{n} y_i dx_i$. What we do here is (i) partition the domain(a,b) into n equal intervals identified by, say i=1,2,3,…,n; (ii) take the value of $y_i=f(x_i)$ at the initial (starting) point of every interval (iii) multiply it by the forward increment dx_i i.e. the forward length of the interval (this gives us the area of the i^{th} strip (iv) sum up all these n areas and finally (v) take the limit as $n\to\infty$. This gives us the area of the region enclosed by the x axis and the curve between the ordinates x=a and x=b.

The relevant point in this procedure is that we take the value of y at the beginning of each partition and multiply it with the forward increment i.e. dx in the direction of increasing x. This is the Ito assumption. y_i is the initial value of y and dx_i was the increment in the direction of increasing x.



Suppose PQR is the curve, PC constitutes my y_i and I multiply with dx_i which is the forward increment CD i.e. y_i is the initial value relevant to the interval CD and CD constitutes the forward difference. This is called the Ito assumption.

It must be emphasized that this scheme is not unique and one can develop a calculus using the final value QD and the backward increment DC or, indeed, a mid-value of y between C & D. All the approaches would be correct. But equations that we arrive at for describing a particular phenomenon would be different, yet it would be equally well described in that framework.

However, this assumption is very important. If we use this assumption, we can do some simplification. g(x) and z become independent of each other and therefore when we take expectations, the expectations get uncoupled and we have e.g.

$$E\left[g(x)z\right]=E\left[g(x)\right]E(z)=0 \text{ since } E(z)=0$$

Similarly,

$E\left[g(x)^2 z^2\right]=E\left[g(x)^2\right]E\left(z^2\right)=E\left[g(x)^2\right] \text{ since } E\left(z^2\right)=1$

Hence, we have:

$$\frac{\partial G(x(t))}{\partial t}=\lim_{dt\to 0}\frac{1}{dt}\left[G'(x)f(x)dt+G'(x)g(x)z\sqrt{dt}+\frac{G''(x)}{2}g(x)^2 z^2 dt+O\left((dt)^{3/2}\right)\right]$$

$$E\left[\frac{\partial G(x(t))}{\partial t}\right]=E\left[G'(x)f(x)+\frac{G''(x)}{2}g(x)^2\right]=E\left[G'(x)f(x)\right]+\frac{1}{2}E\left[G''(x)g(x)^2\right]$$

Now, the expectation of the function F(x(t)) of a random variable x(t) with a probability distribution P(x,t) is defined by:

$E\left[F(x(t))\right]=\int d\omega F(\omega)P(\omega,t)$ so that

$$E\left[\frac{\partial G(x(t))}{\partial t}\right]=\frac{\partial}{\partial t}E\left[G(x(t))\right]=\frac{\partial}{\partial t}\int d\omega\left[G(\omega)P(\omega,t)\right]$$

$$E\left[G'(x)f(x)\right]=\int d\omega G'(\omega)f(\omega)P(\omega,t)$$

$$E\left[G''(x)g(x)^2\right]=\int d\omega G''(\omega)g(\omega)^2 P(\omega,t)$$

Now, G(x) is an arbitrary function. We choose $G(x(t))=\delta(x(t)-X)$. This will have a spike at x(t)=X and will be zero elsewhere. In other words when we do the integration with this delta function in the integrand, it captures only those values for which x=X and throws out the rest of the values of the domain. Consider the integration:

$$E\left[\frac{\partial\delta(x(t)-X)}{\partial t}\right]=\frac{\partial}{\partial t}\int d\omega\left[\delta(\omega-X)P(\omega,t)\right]=\frac{\partial P(X,t)}{\partial t}$$

When we do this integration, this $\delta(\omega-X)$ will operate to pick out $\omega=X$ in the integrand and return $\frac{\partial P(X,t)}{\partial t}$. Now,

$$E\left[f(x)\delta'(x(t)-X)\right]=\int d\omega f(\omega)P(\omega,t)\frac{d}{d\omega}\left[\delta(\omega-X)\right]$$

$$=-\int d\omega\left\{\frac{d}{d\omega}\left[f(\omega)P(\omega,t)\right]\right\}\left[\delta(\omega-X)\right]=-\frac{\partial}{\partial X}\left[f(X)P(X,t)\right]$$

where an integration by parts enables us to shift differentiation from $\delta(\omega-X)$ to $\left[f(\omega)P(\omega,t)\right]$ together with a change in sign, assuming vanishing boundary term. We then use the property of the delta function and obtain the result. For the remaining term we have,

$$E\left[g(x)^2\delta''(x(t)-X)\right]=\int d\omega g(\omega)^2 P(\omega,t)\frac{d^2}{d\omega^2}\left[\delta(\omega-X)\right]$$

$$= \int d\omega \frac{d^2}{d\omega^2}\left[g(\omega)^2 P(\omega,t)\right]\left[\delta(\omega - X)\right] = \frac{\partial^2}{\partial X^2}\left[g(X)^2 P(X,t)\right]$$

we obtain the result by integrating by parts twice and then using the delta function.

Putting all the results together in:

$$E\left[\frac{\partial G(x(t))}{\partial t}\right] = E\left[G'(x) f(x)\right] + \frac{1}{2} E\left[G''(x) g(x)^2\right]$$

we obtain the Fokker Planck equation as:

$$\frac{\partial P(X,t)}{\partial t} = -\frac{\partial}{\partial X}\left[f(X) P(X,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial X^2}\left[g(X)^2 P(X,t)\right]$$

<u>Fokker Planck equation for stock price</u>

The Fokker Planck equation:

$$\frac{\partial P(X,t)}{\partial t} = -\frac{\partial}{\partial X}\left[f(X) P(X,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial X^2}\left[g(X)^2 P(X,t)\right]$$

corresponds to the Langevin equation:

$$dx(t) = f(x)dt + g(x)dW(t)$$

However, our stock price follows the Langevin equation:

$$dS = \mu S dt + \sigma S dW(t)$$

It follows that the Fokker Planck equation for the stock price is:

$$\frac{\partial}{\partial t} p\left(S,t \middle| S',t'\right) = -\frac{\partial}{\partial S}\left[\mu S(t)\, p\left(S,t \middle| S',t'\right)\right] + \frac{1}{2}\frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2(t)\, p\left(S,t \middle| S',t'\right)\right]$$

<u>Solution of the Fokker Planck equation for stock price</u>

The Fokker Planck equation for the stock price is:

$$\frac{\partial}{\partial t} p = -\frac{\partial}{\partial S}\left[\mu S(t)\, p\right] + \frac{1}{2}\frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2(t)\, p\right] \; where \; p = p\left(S,t \middle| S',t'\right)$$ or

$$\frac{\partial}{\partial t} p = \left(\sigma^2 - \mu\right) p + \left(2\sigma^2 - \mu\right) S \frac{\partial p}{\partial S} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 p}{\partial S^2}$$ with boundary conditions:

$$t = t' : p\left(S,t \middle| S',t'\right) = \delta\left(S - S'\right);\; S = 0 : p\left(0,t \middle| S',t'\right) = 0;\; S \to \infty : p\left(S,t \middle| S',t'\right) \to 0$$

The boundary condition are justified as follows:

(i) At t=t’, stock price S=S’;

(ii) if the stock price vanishes at any time, it stays zero thereafter and if S(0)>0, then it can neverbecome zero at any later time so that p(0,t|S’,t’)=0 essentially for S=0;

)iii) price cannot increase unboundedly in a finite interval.

To proceed with the solution, transform variables as follows:

$p=\frac{1}{S'}f(x,\tau);\ln\frac{S}{S'}=x;\ t=t'+\frac{\tau}{\left(\sigma^2/2\right)}$ whence we get

$\frac{\partial f}{\partial\tau}=\frac{\partial^2 f}{\partial x^2}+[3-k]\frac{\partial f}{\partial x}+[2-k]f\ \ \textit{with}\ k=\frac{2\mu}{\sigma^2}$ with transformed boundary conditions:

$\tau=0: f(x,0)=\delta\left(e^x-1\right); x\to\pm\infty: f(x,\tau)\to 0$

To convert the above equation to a diffusion equation, we make a second substitution: $f(x,\tau)=e^{\alpha x+\beta\tau}g(x,\tau)$ where α, β are free variables. We get the transformed equation as:

$$\frac{\partial g}{\partial\tau}=\frac{\partial^2 g}{\partial x^2}+\left[2\alpha+(3-k)\right]\frac{\partial g}{\partial x}+\left[\alpha^2+(3-k)\alpha+2-k-\beta\right]g$$

We can, now set $\alpha=\frac{1}{2}(k-3);\ \beta=\alpha^2+(3-k)\alpha+2-k=-\frac{1}{4}(k-1)^2$ to transform the above equation to the diffusion equation:

$\left(\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial x^2}\right)g(x,\tau)=0$ with the boundary conditions:

$\tau=0: g(x,0)=e^{-(k-3)x/2}\delta\left(e^x-1\right)\Rightarrow g(x,0)e^{-\alpha x^2}\xrightarrow{|x|\to\infty}0(\alpha>0)$ and

$\tau>0: g(x,\tau)e^{-\alpha x^2}\xrightarrow{|x|\to\infty}0(\alpha>0)$

which has the standard solution:

$$g(x,\tau)=\frac{1}{\sqrt{4\pi\tau}}\exp\left(-\frac{x^2}{4\tau}\right).$$

On retracing tto the original variables, we get:

$$f(x,\tau)=\frac{1}{\sqrt{4\pi\tau}}\exp\left[\frac{1}{2}(k-3)x-\frac{1}{4}(k-1)^2\tau\right]\exp\left(-\frac{x^2}{4\tau}\right)$$
$$=\frac{e^{-x}}{\sqrt{4\pi\tau}}\exp\left\{-\frac{\left[x-(k-1)\tau\right]^2}{4\tau}\right\}.$$

Using $k = \frac{2\mu}{\sigma^2}$ so that $(k-1)\tau = \left(\mu - \frac{\sigma^2}{2}\right)(t-t')$, we have, finally,

$$p(S,t|S',t') = \frac{1}{\sqrt{2\pi(\sigma S)^2(t-t')}} \exp\left\{ -\frac{\left[\ln\left(\frac{S}{S'}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(t-t')\right]^2}{2\sigma^2(t-t')} \right\}$$

Setting $\xi = \ln S$

$$p(\xi,t|\xi',t') = \frac{1}{\sqrt{2\pi\sigma^2(t-t')}} \exp\left\{ -\frac{\left[(\xi-\xi') - \left(\mu - \frac{\sigma^2}{2}\right)(t-t')\right]^2}{2\sigma^2(t-t')} \right\}$$

This is clearly the PDF of a normal distribution with a mean of $\left(\mu - \frac{\sigma^2}{2}\right)(t-t')$ and a variance of $\sigma^2(t-t')$. Thus, $\xi = \ln S$ is normally distributed as above implying that S is lognormally distributed.

Appendix

Some properties of lognormal distributions

The lognormal probability density function

Let X be a random variable following the lognormal distribution with PDF ρ(x). Then, Y=lnX is normally distributed, say with parameters μ and σ^2.

Then, we have PDF of Y as:

$$p(y) = \frac{1}{\sqrt{2\pi\ \sigma^2}} \mathbf{exp}\left[-\frac{(y-\mu)^2}{2\sigma^2} \right]$$

Now, by the definition of PDF, we have:

$$P(x < X < x + dx) = \rho(x)dx = P(\mathbf{ln}\,x < \mathbf{ln}\,X < \mathbf{ln}(x+dx))$$
$$= P(y < Y < y + dy) = p(y)dy = p(\mathbf{ln}\,x)[\mathbf{ln}(x+dx) - \mathbf{ln}\,x]$$
$$= p(\mathbf{ln}\,x)\left[\mathbf{ln}\left(1+\frac{dx}{x}\right)\right] = p(\mathbf{ln}\,x)\frac{dx}{x} = \frac{dx}{x\sqrt{2\pi\ \sigma^2}} \mathbf{exp}\left[-\frac{(\mathbf{ln}\,x-\mu)^2}{2\sigma^2} \right]$$

Mean of lognormal distribution

$$E(X) = \int_0^\infty xf(x)dx = \int_0^\infty x\frac{1}{x}\frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2} \right] dx$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{(u-\mu)^2}{2\sigma^2} \right] e^u du \qquad (Put\ u = \ln x)$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left\{ -\left[\frac{(u-\mu)^2 - 2u\sigma^2}{2\sigma^2} \right]\right\} du = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left\{ -\left[\frac{(u-\mu-\sigma^2)^2 - 2\mu\sigma^2 - \sigma^4}{2\sigma^2} \right]\right\} du$$
$$= \exp\left(\frac{2\mu\sigma^2 + \sigma^4}{2\sigma^2} \right)\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left\{ -\left[\frac{(u-\mu-\sigma^2)^2}{2\sigma^2} \right]\right\} du = \exp\left(\mu + \frac{\sigma^2}{2} \right)$$

The last step follows because $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left\{ -\left[\frac{(u-\mu-\sigma^2)^2}{2\sigma^2} \right]\right\} du = 1$ since $\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\left[\frac{(u-\mu-\sigma^2)^2}{2\sigma^2} \right]\right\} du$ is the PDF of a normal variate with a mean of $\mu + \sigma^2$ and a standard deviation of σ.

Variance of lognormal distribution

We have, $E(X^2) = \int_0^\infty x^2 f(x)dx = \int_0^\infty x \frac{1}{\sqrt{2\pi}\sigma} \exp\left[- \frac{(\ln x - \mu)^2}{2\sigma^2} \right] dx$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^u \exp\left[- \frac{(u-\mu)^2}{2\sigma^2} \right] e^u du = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{ - \left[\frac{(u-\mu)^2 - 4u\sigma^2}{2\sigma^2} \right] \right\} du$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{ - \left[\frac{(u-\mu-2\sigma^2)^2}{2\sigma^2} - \frac{4\mu\sigma^2 + 4\sigma^4}{2\sigma^2} \right] \right\} du$$

$$= \exp\left(\frac{4\mu\sigma^2 + 4\sigma^4}{2\sigma^2} \right) \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left[- \frac{(u-\mu-2\sigma^2)^2}{2\sigma^2} \right] du = \exp(2\mu + 2\sigma^2)$$

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \exp(2\mu + \sigma^2)\left(\exp(\sigma^2) - 1\right)$$

$$E\left(\frac{S_t}{S_0} \right) = \exp(\mu t), \textit{ or } E(S_t) = S_0 \exp(\mu t)$$

$$Var\left(\frac{S_t}{S_0} \right) = \exp(2\mu t)\left(\exp(\sigma^2 t) - 1\right) \textit{ or } Var(S_t) = S_0^2 \exp(2\mu t)\left(\exp(\sigma^2 t) - 1\right)$$