Financial Derivatives & Risk Management
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Lecture 40 - Stochastic Processes: Central Limit Theorem, Stochastic Calculus

Solution of the diffusion equation

We shall solve the diffusion equation:

$$\frac{\partial P(X,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 P(X,t)}{\partial X^2}$$

using the Fourier transform. The Fourier transform of a given function f(x) of x is defined as $\hat{f}(k)$ where:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

with the inverse Fourier transform being given by:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk .$$

We take the Fourier transform of both the sides of the diffusion equation. On taking the Fourier transform of the LHS, the derivative is with respect to t and the integral with respect to x, so we can take the derivative outside the integration without any problem and whatever remains inside the integral is nothing but the Fourier transform of P i.e.

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} P(x,t) e^{-ikx} dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} P(x,t) e^{-ikx} dx = \frac{\partial}{\partial t}\hat{P}(k,t)$$

For the RHS, we have:

Integrating by parts, we have:

$$\frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\partial^2}{\partial x^2}P(x,t)\right]e^{-ikx}dx=\left(\frac{1}{2}\frac{\partial}{\partial x}P(x,t)e^{-ikx}\right)\Bigg|_{-\infty}^{+\infty}+\frac{ik}{2}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial x}P(x,t)\right]e^{-ikx}dx$$

Assuming that the boundary term vanishes sufficiently fast, we have:

$$\frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\partial^2}{\partial x^2}P(x,t)\right]e^{-ikx}dx=\frac{ik}{2}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial x}P(x,t)\right]e^{-ikx}dx$$

Another integration by parts yields:

$$\frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\partial^2}{\partial x^2}P(x,t)\right]e^{-ikx}dx=\left(\frac{ik}{2}P(x,t)e^{-ikx}\right)\Bigg|_{-\infty}^{+\infty}-\frac{k^2}{2}\int_{-\infty}^{\infty}P(x,t)e^{-ikx}dx$$

whence on assuming that the boundary term does not contribute, we get:

$$\frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\partial^2}{\partial x^2}P(x,t)\right]e^{-ikx}dx=-\frac{k^2}{2}\int_{-\infty}^{\infty}P(x,t)e^{-ikx}dx=-\frac{k^2}{2}\hat{P}(k,t)$$

so that the Fourier transform of the diffusion equation yields:

$$\frac{\partial\hat{P}(k,t)}{\partial t}=-\frac{k^2}{2}\hat{P}(k,t)\ or\ \hat{P}(k,t)=\hat{P}(k,0)e^{-\frac{1}{2}k^2t}$$

The initial condition is P(x,0)=δ(x). Now, the Fourier transform of δ(x)=1 since

$$\hat{\delta}(k)=\int_{-\infty}^{\infty}\delta(x)e^{-ikx}dx=e^0=1$$

so that $\hat{P}(k,0)=\hat{\delta}(k)=1$ whence,

$$\hat{P}(k,t)=e^{-\frac{1}{2}k^2t}$$

Now, we take the inverse Fourier transform, whence we get:

$$P(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\frac{1}{2}k^2t+ikx}dk=\frac{1}{2\pi}e^{-\frac{x^2}{2t}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\left(k\sqrt{t}-i\frac{x}{\sqrt{t}}\right)^2}dk$$

$$=\frac{1}{2\pi\sqrt{t}}e^{-\frac{x^2}{2t}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\left(y-i\frac{x}{\sqrt{t}}\right)^2}dy=\frac{1}{2\pi\sqrt{t}}e^{-\frac{x^2}{2t}}\sqrt{2\pi}=\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$$

This is clearly the PDF of a normal distribution with a mean of 0 and with a variance of t, so that establishes that the variable that we were talking about is normally distributed with a mean of 0 and a variance of t, which is nothing but Brownian motion precisely what we wanted to establish.

Central Limit Theorem: An Illustration

I will not go into the proof of the CLT but, nevertheless, I shall illustrate it with an illuminating example that explicitly propounds its cardinal property viz that the underlying distributions of the variables is irrelevant. The limiting distribution invariably approaches the normal distribution irrespective of the underlying distributions of the random variables themselves. The only requirement is the finiteness of the means and variances of these underlying distributions. The theorem states that:

Let X_i; i=1,2,…,*n* be independent identically distributed random variables each with finite mean and variance μ and σ^2 respectively. Then the following expression is distributed as a standard normal variate.

$$Z_n = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}}$$

To illustrate the theorem, we consider X_i, i=1,2,..,*n* as uniformly distributed IIDs over the interval (0,1) Then, we have:

$$1 = \int_0^1 p(x_i)\,dx_i = p\int_0^1 dx_i = p = p(x_i)$$

so that $p(x_i)=1$, $\forall x_i \in [0,1]$, i=1,2,…,*n*.

$$\mu_i = E(X_i) = \int_0^1 x_i\,p(x_i)\,dx_i = \int_0^1 x_i.1.dx_i = \frac{1}{2}$$

$$E(X_i^2) = \int_0^1 x_i^2\,p(x_i)\,dx_i = \int_0^1 x_i^2.1.dx_i = \frac{1}{3}$$

$$\sigma_i^2 = Var(X_i) = E(X_i^2) - \left[E(X_i)\right]^2 = \frac{1}{12}$$

$$Define\ Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} = \frac{\sum_{i=1}^{n} X_i - \frac{n}{2}}{\sqrt{n/12}}$$

$$The\ pdf\ of\ Z_n\ is$$

$$\rho(z_n) = \int_0^1 p(x_1)\,dx_1 \ldots \int_0^1 p(x_n)\,dx_n\,\delta(z - z_n)$$

$$But\ pdf\ of\ each\ X_i\ is\ unity\ so$$

$$\rho(z_n) = \int_0^1 dx_1 \ldots \int_0^1 dx_n\,\delta(z - z_n)\ \ where\ z_n = \frac{\sum_{i=1}^{n} x_i - \frac{n}{2}}{\sqrt{n/12}}$$

Please note the difference between Z_n and z_n and X_i and x_i. The capital letter e.g. Z_n represent the random variable itself while the small letter e.g. z_n is a value that the random variable Z_n could possibly take from its sample set.

The delta function $\delta(z - z_n)$ constraint has been introduced into the integral so that, when integrating with respect to z, only the values of Z_n given by $z_n = \frac{\sum_{i=1}^{n} x_i - \frac{n}{2}}{\sqrt{n/12}}$ contribute to the integral. In other words, only those points will contribute to the integral i.e. to this probability density function $\rho(z_n)$ where $z_n = \frac{\sum_{i=1}^{n} x_i - \frac{n}{2}}{\sqrt{n/12}}$ is fulfilled. Now,

$$\rho_n(z) = \int_0^1 dx_1 ... \int_0^1 dx_n \delta(z - z_n)$$

Using the Fourier representation of δ function:

$$\delta(z - z_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp[ik(z - z_n)]$$

$$\rho(z) = \frac{1}{2\pi} \int_0^1 dx_1 ... \int_0^1 dx_n \int_{-\infty}^{\infty} dk \exp[ik(z - z_n)]$$

$$= \frac{1}{2\pi} \int_0^1 dx_1 ... \int_0^1 dx_n \int_{-\infty}^{\infty} dk \exp\left\{ik\left[z - \left(\frac{\sum_{i=1}^{n} x_i - \frac{n}{2}}{\sqrt{n/12}}\right)\right]\right\}$$

$$= \frac{1}{2\pi} \int_0^1 dx_1 ... \int_0^1 dx_n \int_{-\infty}^{\infty} dk \exp\left\{ik\left[\left(z + \sqrt{3n}\right) - \left(\frac{\sum_{i=1}^{n} x_i}{\sqrt{n/12}}\right)\right]\right\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp\left[ik\left(z + \sqrt{3n}\right)\right] \int_0^1 \exp\left(-\frac{ikx_1}{\sqrt{n/12}}\right) dx_1 ... \int_0^1 \exp\left(-\frac{ikx_n}{\sqrt{n/12}}\right) dx_n$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp\left[ik\left(z + \sqrt{3n}\right)\right] \left(\int_0^1 \exp\left(-\frac{ikx}{\sqrt{n/12}}\right) dx\right)^n$$

$$I = \int_0^1 \exp\left(-\frac{ikx}{\sqrt{n/12}}\right) dx = \frac{\sqrt{n/12}}{-ik}\left(e^{-ik\frac{1}{\sqrt{n/12}}} - 1\right) = \frac{\sqrt{n/3}}{-k} e^{-ik\frac{1}{\sqrt{n/3}}} \times$$

$$\left(\frac{e^{-ik\frac{1}{\sqrt{n/3}}} - e^{ik\frac{1}{\sqrt{n/3}}}}{2i}\right) = \frac{\sqrt{n/3}}{k} e^{-ik\frac{1}{\sqrt{n/3}}} \sin\left(k\frac{1}{\sqrt{n/3}}\right) = e^{-ik\frac{1}{\sqrt{n/3}}} \frac{\sin\left(k\frac{1}{\sqrt{n/3}}\right)}{k\frac{1}{\sqrt{n/3}}}$$

$$\rho(z_n)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ik\left(z+\sqrt{3n}\right)}\left[e^{-ik\frac{1}{\sqrt{n/3}}}\frac{\sin\left(k\frac{1}{\sqrt{n/3}}\right)}{k\frac{1}{\sqrt{n/3}}}\right]^n=\frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ikz}\left[\frac{\sin\left(k\frac{1}{\sqrt{n/3}}\right)}{k\frac{1}{\sqrt{n/3}}}\right]^n$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ikz}\left[1-\frac{1}{3!}\left(k\frac{1}{\sqrt{n/3}}\right)^2\right]^n=\frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ikz}\left(1-\frac{3k^2}{6n}\right)^n.$$

$$Now, \lim_{n\to\infty}\left(1-\frac{3k^2}{6n}\right)^n=e^{-k^2/2}\ \ so\ that\ \ \rho(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dke^{ikz-k^2/2}$$

$$=\frac{1}{2\pi}e^{-\frac{1}{2}z^2}\int_{-\infty}^{\infty} dke^{\frac{1}{2}(ik+z)^2}=\frac{1}{2\pi}e^{-\frac{1}{2}z^2}\left(\sqrt{2\pi}\right)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$$

$$Put\ \left(ik+z\right)=i\theta;\ dk=d\theta$$

This is nothing but the PDF of the standard normal variate. So, after a lot of manipulation and we end up with the standard normal variate. We started with n uniformly distributed variables in 0, 1 and we ended with the normal distribution.

So, this is the beauty of the central limit theorem, the central limit theorem does not depend on the distribution of the underlying variables which are being added to form the variable whose distribution is being considered. The underlying distribution is irrelevant. The requirements are (i) the mean and variance should be finite and (ii) the variables should be IIDs i.e. independent identically distributed variables.

Brownian motion with drift

The normal scaling of the n-step random walk that leads to BM is √(T/n). However, by introducing a scaling of the step size as σ√(T/n), we are able to adjust or manipulate the dispersion or volatility of the BM to conform to the data which we desire to model using the BM framework. By introducing σ into the scaling of jump size, we are able to adapt the concept of Brownian motion to fit in variety of data. In fact, not only this, we can also add a trend or drift to the BM by using a step size of μ(T/n)+σ√(T/n) where μ is the rate of drift i.e. drift per unit time. We, then, have a combination of a non-stochastic term and a stochastic term. The stochastic term is scaled by σ and the non-stochastic term by μ. The corresponding BM is called generalized Brownian motion or Brownian motion with drift. The infinitesimal increment of a generalized Wiener process can be expressed as:

dx = μdt + σdW

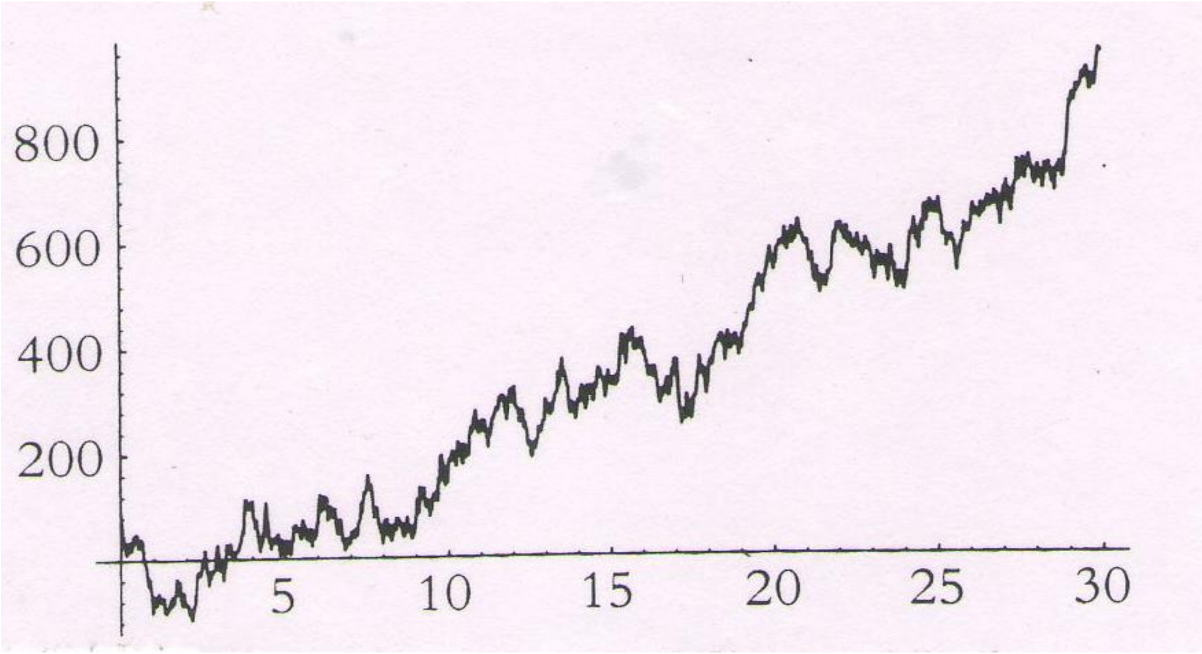
It has two components:

(i) a non-stochastic component given by μdt representing the drift and
(ii) a stochastic component given by σdW where dW = z√dt, z is N(0,1) and σ is a scaling factor of the dispersion or diffusion of the process.

Mean & variance of generalized BM

For the generalized Wiener process:

$dx = \mu dt + \sigma dW$
Mean = μdt;   Variance = $\sigma^2 dt$



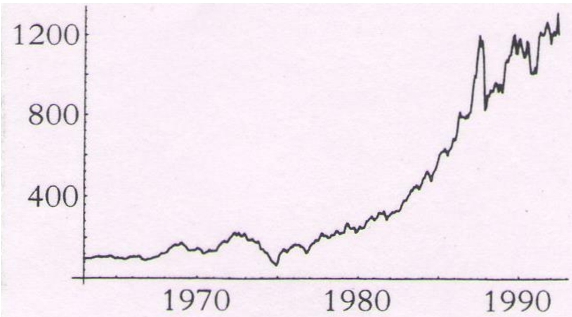
A clearly perceptible upward bias is discernible in the above diagram. The depiction shows an obvious random superposition over an underlying upwardly drifted process. The Brownian motion is random, it is fluctuating but it is fluctuating with a clearly perceptible upward bias and that is what is called a Brownian motion with drift. Of course the drift can also be negative nobody prevents there being a negatively biased Brownian motion.

Drift rate & diffusion rate

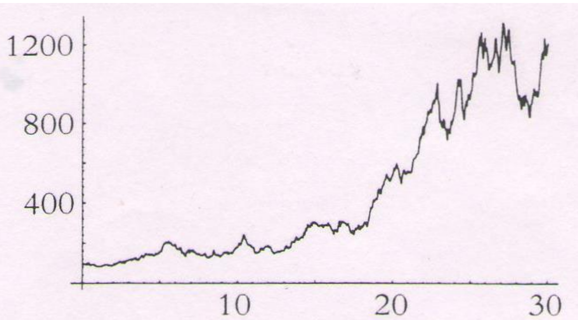
The mean drift per unit time for a stochastic process is called the drift rate.
The variance per unit time is called the variance rate.

Exponential Brownian motion

$$\mathbf{X_t = X_0 \exp\left(\mu t + \sigma W(t)\right)}$$



UK FTA index, 1963–92



Exponential Brownian motion

We also have exponential Brownian motion. There is an underlying exponential curve and BM is superposed over that exponential curve. Thus, random fluctuations manifest themselves about an exponential curve. The zigzag is there but the zigzag is superposed on a certain exponential curve.

Ito process

In the case of generalized BM, we had $dx=\mu dt+\sigma dW$. In this case, μ and σ were constants, independent of x & t. But there exist stochastic processes where both these quantities μ and σ are functions of x and t. These are caused Ito processes where the drift as well as the diffusion or the variance are functions of x and t. The infinitesimal increment of an Ito process can be expressed as:

$$dx = a(x,t)dt + b(x,t)dW$$

Example 1

A particle starts executing standard Brownian motion at t=0. What is the probability that the particle will be less than 10 units away from the origin after 100 units of time?

Solution

The spectrum of possible values of a BM W(t) at time t after initiation is normally distributed with a mean of 0 and a variance of t i.e. a standard deviation of $\sqrt{(t)}$.

Hence, the possible values after time t=100 are normally distributed with a mean of 0 and standard deviation of $\sqrt{(100)} = 10$.

We need to find the probability that $-10<W(100)<+10$ i.e. $P(-10<X<+10)$ where $W(100)=X$ is $N(0,100)$.

Expressing the above in terms of the standard normal variate $Z=\dfrac{X-\mu}{\sigma}$ we need to find out:

$$P(-10 < X < +10) = P\left(\frac{-10-0}{10} < Z < \frac{10-0}{10}\right) = P(-1 < Z < +1) = 0.6826$$

where the value has been obtained from normal tables.

Example 2

A particle executes scaled Brownian motion with drift in one dimension. The drift rate per unit time is 0.0001 units. The variance rate is 0.01. Calculate the probability that the particle is more than 2 units on the positive side of its initial position (origin) after 2500 units of time.

Solution

The spectrum of possible values of a BM with drift i.e.

$$dx=\mu dt+\sigma dW_t=\mu dt+\sigma z\sqrt{(dt)}$$

at time dt after initiation is normally distributed with a mean of μdt and a variance of $\sigma^2 dt$ i.e. a standard deviation of $\sigma\sqrt{(dt)}$.

Here $\mu=0.0001$, $\sigma^2=0.01$ and t=2500.

Hence, the possible values after time t=2500 are normally distributed with a mean of 0.25 and standard deviation of $\sqrt{(0.01 *2500)} =5$.

Thus, after 2500 units of time, the mean position of the particle (as shown above) is 0.0001*2500=0.25 above the origin and the variance of the possible positions of the particle is 25 i.e. standard deviation is 5.
We need to find the probability of the particle being more than 2 units above the origin after 2500 units of time.

Thus, we need to find out $P(X>2.00)$ where X is N(0.25,25)

Converting to standard normal variate $P(X>2.00)=P[Z>(2.00-0.25)/5]=P(Z>0.35)=0.3632$

where the value has been obtained from normal tables.

Example 3

The marks in a class are believed to follow a normal distribution with a mean of 60 and a variance of 144. If the total number of students in the class is 100, what is the number of students who have obtained less than 42 marks?

Solution

Let X represent the marks obtained by various students in the class. Then it is given that X is N(60,144). We need to find $P(X<42)$.

The standard normal variate Z corresponding to X=42 is $(42-60)/12 = -1.5$

Hence, we need to find $P(Z<-1.5)=0.5000-0.3531=0.1469$

Since there are 100 students in the class, number of students below 42 =14.69 i.e. 14 students.

Appendix

Linear Combination of Random Variables

Let X,Y be random variables and let W=αX+βY where α, β are fixed (not random) real numbers. We need to find the mean and variance of W. We have

E(W)=E(αX+βY)= E(αX)+E(βY)= αE(X)+βE(Y)

Also $E(W^2)= E(\alpha X+\beta Y)^2= E(\alpha^2X^2+\beta^2Y^2+2\alpha\beta XY)= \alpha^2E(X^2)+\beta^2E(Y^2)+2\alpha\beta E(XY)$

$[E(W)]^2=\alpha^2[E(X)]^2+\beta^2[E(Y)]^2+2\alpha\beta E(X)E(Y)$

$Var(W)=E(W^2)- [E(W)]^2=\alpha^2\{E(X^2)- [E(X)]^2\}+\beta^2\{E(Y^2)- [E(Y)]^2\}+2\alpha\beta\{E(XY)-E(X)E(Y)\}$

$=\alpha^2Var(X)+\beta^2Var(Y) +2\alpha\beta Cov(X,Y)$

$=\alpha^2Var(X)+\beta^2Var(Y) +2\alpha\beta CORREL(X,Y)SD(X)SD(Y)$

For a linear combination of n-random variables $W=\sum_{i=1}^{n}\alpha_i X_i$, we have: $E(W)=\sum_{i=1}^{n}\alpha_i E(X_i)$,

$$\sigma_W^2=\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\sigma_{ij}=\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\rho_{ij}\sigma_i\sigma_j=\sum_{i=1}^{n}\alpha_i^2\sigma_i^2+\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\alpha_i\alpha_j\sigma_{ij}$$

$$=\sum_{i=1}^{n}\alpha_i^2\sigma_i^2+\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\alpha_i\alpha_j\rho_{ij}\sigma_i\sigma_j$$

$$=\sum_{i=1}^{n}\alpha_i^2\sigma_i^2+2\sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n}\alpha_i\alpha_j\sigma_{ij}=\sum_{i=1}^{n}\alpha_i^2\sigma_i^2+2\sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n}\alpha_i\alpha_j\rho_{ij}\sigma_i\sigma_j$$

Some Results on Normal Distribution

Linear combination of normally distributed variables

If $X_1,\dots,X_n$ are independent normal variables, X_i being $N(\mu_i,\sigma_i^2)$ then the variate $W=\sum_{i=1}^{n}\alpha_i X_i$ *is a normally distributed* $N(\mu',\sigma^2)$ *random variable with*

$$\mu'=\sum_{i=1}^{n}\alpha_i\mu_i \text{ and } \sigma=\sqrt{\sum_{i=1}^{n}\alpha_i^2\sigma_i^2}$$

where α_i's are constants.

Proof

The characteristic function of W is

$$\psi_W(t)=E\left(e^{itW}\right)=E\left(e^{it\sum_{i=1}^{n}\alpha_i X_i}\right)=E\left\{\prod_{i=1}^{n}e^{it\alpha_i X_i}\right\}=\prod_{i=1}^{n}E\left(e^{it\alpha_i X_i}\right)$$

(by virtue of independence of the X_i variates)

$$= \prod_{i=1}^{n} \psi_{\alpha_i X_i(t)} = \prod_{i=1}^{n} \left(e^{\alpha_i \mu_i t - \frac{1}{2} t^2 \sigma_i^2 \alpha_i^2} \right) = e^{t\mu' - \frac{1}{2} t^2 \sigma'^2}$$

which is the characteristic functions of an N(μ’,σ’) distributed random variable.

Hence the result follows by the uniqueness property of characteristic function.

It follows as a corollary that if $X_1,\ldots,X_n$ are independent normal variables, X_i being $N(\mu_i,\sigma_i^2)$. Then the sum

$$S_n = \sum_{i=1}^{n} X_i \ is\ an\ N\left(\mu', \sigma'^2\right) \text{ variable where } \mu' = \sum_{i=1}^{n} \mu_i \ and\ \sigma' = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

The normal distribution PDF from binomial PDF

Let X be a binomial variate with mean np and variance npq. If n → ∞, the probability distribution of the standardized binomial variable $z = \frac{(X - np)}{\sqrt{npq}}$ tends to that of a unit normal variate.

Proof

From the binomial distribution, we have: $P(X = r) = P(n,r) = \frac{n!}{r!(n-r)!} p^r q^{n-r}$

Applying Stirling’s approximation to factorials viz.

$\Gamma(n+1) = n! = \sqrt{2\pi} n^{\left(n+\frac{1}{2}\right)} e^{-\left(n+\frac{\theta_n}{12n}\right)}$ $where\ 0 < \theta_n < 1$, we get, using $\theta_n \cong 0$

$$\lim_{n\to\infty} P(n,r) = \frac{e^{-n} n^{n+\frac{1}{2}} p^r q^{n-r}}{\sqrt{2\pi} e^{-r} r^{r+\frac{1}{2}} e^{-(n-r)} (n-r)^{n-r+\frac{1}{2}}} = \frac{A}{\sqrt{2\pi npq}} (say)$$

where $A = \left(\frac{np}{r}\right)^{r+\frac{1}{2}} \left(\frac{nq}{n-r}\right)^{n-r+\frac{1}{2}} = \left(1 + z_r \sqrt{\frac{q}{np}}\right)^{-\left(np + z_r\sqrt{npq} + \frac{1}{2}\right)} \left(1 - z_r \sqrt{\frac{p}{nq}}\right)^{-\left(nq - z_r\sqrt{npq} + \frac{1}{2}\right)}$

where $z_r = \frac{(r - np)}{\sqrt{npq}}$. Expanding the logarithm, we get

$$\log_e A = -\left(np + z_r\sqrt{npq} + \frac{1}{2}\right)\log_e\left(1 + z_r\sqrt{\frac{q}{np}}\right) - \left(nq - z_r\sqrt{npq} + \frac{1}{2}\right)\log_e\left(1 - z_r\sqrt{\frac{p}{nq}}\right)$$

Assuming $|z_r| < \min\left(\sqrt{\frac{np}{q}}, \sqrt{\frac{nq}{p}}\right)$, we have:

$$\log_e A = -\left(np + z_r\sqrt{npq} + \frac{1}{2}\right)\left[z_r\sqrt{\frac{q}{np}} - \frac{z_r^2 q}{2np} + \frac{z_r^2 q^{\frac{3}{2}}}{3(np)^{\frac{3}{2}}} + 0\left(n^{-2}\right)\right]$$

$$+\left(nq - z_r\sqrt{npq} + \frac{1}{2}\right)\left[z_r\sqrt{\frac{p}{nq}} + \frac{z_r^2 p}{2nq} + \frac{z_r^3 p^{\frac{3}{2}}}{3(nq)^{\frac{3}{2}}} + 0\left(n^{-2}\right)\right]$$

$$= \frac{z^2}{2} + \frac{z(q-p)}{2\sqrt{npq}} - \frac{z^2}{4npq}\left(p^2+q^2\right) - \frac{z^3}{6\sqrt{npq}}(q-p) + \frac{z^3}{6(npq)^{\frac{3}{2}}}\left(q^3-p^3\right)$$

As $n \to \infty$, the above expression approaches $\frac{z^2}{2}$

As r takes values 0,1,…,n and n → ∞, z takes values between - ∞ and ∞. Also as r increases by value 1,z increased by the amount $\frac{1}{\sqrt{npq}}$, which for large n, would be taken as dz. Hence z can be taken as a continuous variable taking values between -∞ and +∞ and $\lim_{n\to\infty} P(n,r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z_r^2}{2}} dz, -\infty \le z_r \le \infty$.