Financial Derivatives & Risk Management
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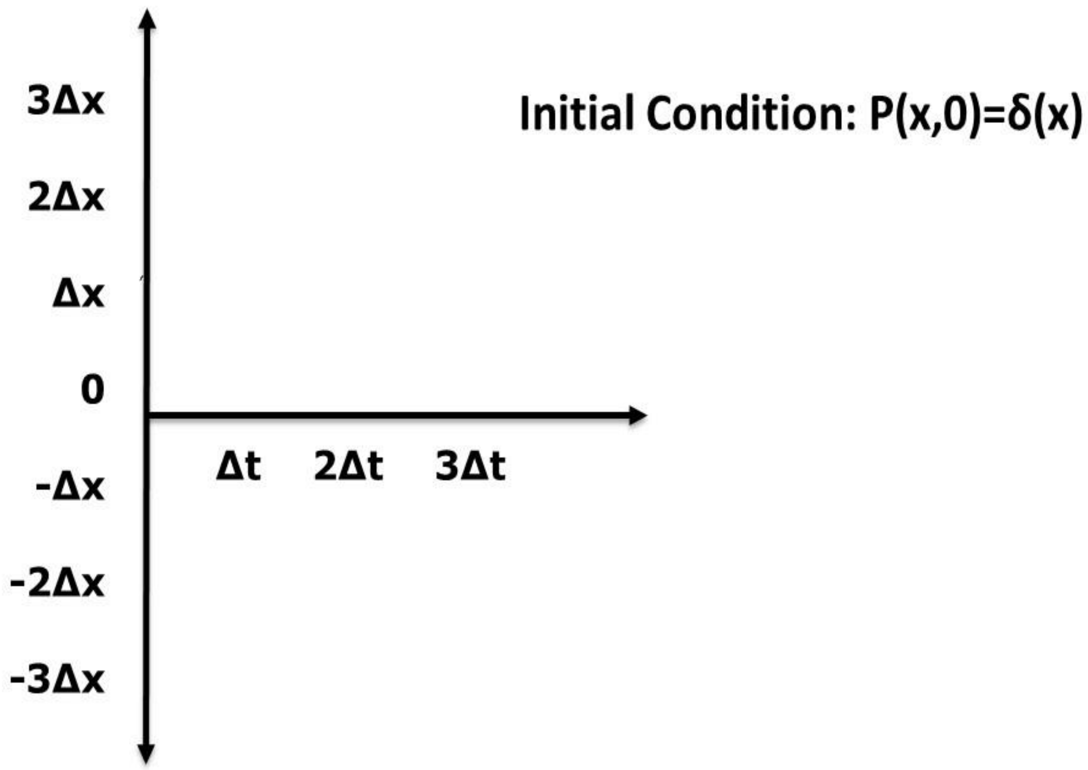
Lecture 39 - Stochastic Processes: Diffusion Equation

Now, as I mentioned earlier, we can arrive at the mathematical structure of Brownian Motion through a diffusion phenomenology. In fact, this approach is more transparent, instructive and intuitive. The very term Brownian motion was coined to commemorate the work of Robert Brown in this area who studied the dynamics of pollen grains subjected to a fluid motion. The grains were immersed in a liquid and their movements minutely studied under a microscope. The resulting patterns were studied. We can start with the concept of diffusion and arrive at arrive at the mathematical formulation of Brownian motion.

Consider a particle executing a random walk on a one-dimensional infinite discrete lattice. Let the length of each step i.e. the distance between two neighboring points on the lattice be Δx. Let each time step be of length Δt. Let the discrete points on the lattice be identified with the coordinates $x = 0, \pm 1\Delta x, \pm 2\Delta x...$.

Let the point be assumed to start at the origin at t=0 so that the initial condition is $P(x,0)=\delta(x)$ at t=0. Let its position at a given instant t=t be at the coordinate x=X.



Let

(i) p=probability that the randomly moving particle moves one step to the right in the next step i.e. the probability that the coordinate increases by Δx in time Δt;

(ii) q=probability that the randomly moving particle moves one step to the left in the next step i.e. the probability that the coordinate decreases by Δx in time Δt;

(iii) r=1-p-q=probability that the randomly moving particle stays in Δt where it is at time t.

(iv) p,q,r are assumed constant over the length of the walk i.e. for all time and space steps.

(v) P(X,t)=probability of finding the particle at the coordinate x=X at time t=t.

The particle is executing random motion and let us say, at any point in time the particle is at a coordinate x=X. x is the general representation of the x axis, x=X is a specific value of small x at which we are exploring the dynamics of the particle. Then, p be the probability that the randomly moving particle moves 1 step to the right. In other words, if it moves from, say, current coordinate 3Δx to 4Δx i.e. in the direction of increasing coordinate x. Similarly, q is the probability that it moves 1 step to the left, that means it moves in the order of decreasing coordinate x. The probability of the particle not moving anywhere is r. p, q and r are assumed to be uniform over the length of the walk i.e. for all time and space steps. P(X,t) is the probability of finding the particle at coordinate x=X at time t.

Now, a particle can be at position x=X at time t+Δt if

(i) the particle is at X-Δx at time t, the probability of which is P(X-Δx,t) and it moves one step of distance Δx to the right in time Δt, the probability of which is p; or

(ii) the particle is at X+Δx at time t, the probability of which is P(X+Δx,t) and it moves one step of length Δx to the left in time Δt, the probability of which is q; or

(iii) the particle is at X at time t, the probability of which is P(X,t) and it does not move in time Δt, the probability of which is r=1-p-q.

Thus,

(i) pP(X-Δx,t) is the probability that the particle will reach x=X at t=t+Δt by moving one step to the right in Δt;

(ii) qP(X+Δx,t) is the probability that the particle will reach x=X at t=t+Δt by moving one step to the left in Δt;

(iii) (1-p-q)P(X,t) is the probability that the particle will remain at x=X at t=t+Δt if it is at x=X at t=t.

Further, we assume that the particle can take only one step in the time interval Δt. So, the total probability of finding the particle at x=X at time t+Δt is the sum of (i)+(ii)+(iii) above i.e.

$$P(X,t+\Delta t) = pP[X-\Delta x,t] + qP[X+\Delta x,t] + (1-p-q)P(X,t) \text{ or}$$
$$P(X,t+\Delta t) - P(X,t) = pP[X-\Delta x,t] + qP[X+\Delta x,t] - (p+q)P(X,t)$$

Now, in the limiting case, the LHS can be written as:

$$P(X,t+\Delta t) - P(X,t) = \Delta t\left[\frac{\partial P(X,t)}{\partial t}\right]$$

Let us look at the RHS

$$pP\left[X-\Delta x,t\right]+qP\left[X+\Delta x,t\right]-\left(p+q\right)P\left(X,t\right)$$
$$=\frac{1}{2}\left(p+q\right)\left\{P\left[X-\Delta x,t\right]-2P\left(X,t\right)+P\left[X+\Delta x,t\right]\right\}$$
$$+\frac{1}{2}\left(p-q\right)\left\{P\left[X-\Delta x,t\right]-P\left[X+\Delta x,t\right]\right\}$$

Now

$$P\left[X+\Delta x,t\right]-P\left[X-\Delta x,t\right]\approx 2\Delta x\frac{\partial P\left(X,t\right)}{\partial X}$$
$$P\left[X+\Delta x,t\right]-P\left(X,t\right)\approx \Delta x\frac{\partial P\left(X,t\right)}{\partial X}$$
$$P\left(X,t\right)-P\left[X-\Delta x,t\right]\approx \Delta x\frac{\partial P\left(X-\Delta x,t\right)}{\partial X}$$
$$P\left[X-\Delta x,t\right]-2P\left(X,t\right)+P\left[X+\Delta x,t\right]$$
$$=\Delta x\left[\frac{\partial P\left(X,t\right)}{\partial X}-\frac{\partial P\left(X-\Delta x,t\right)}{\partial X}\right]\approx\left(\Delta x\right)^2\frac{\partial^2 P\left(X,t\right)}{\partial X^2}$$

Putting the pieces together

$$\frac{\partial P\left(X,t\right)}{\partial t}=\frac{\left(\Delta x\right)^2}{2\Delta t}\left(p+q\right)\frac{\partial^2 P\left(X,t\right)}{\partial X^2}-\frac{\Delta x}{\Delta t}\left(p-q\right)\frac{\partial P\left(X,t\right)}{\partial X}$$

For an unbiased random walk p=q= ½

$$\frac{\partial P\left(X,t\right)}{\partial t}=\frac{\left(\Delta x\right)^2}{2\Delta t}\frac{\partial^2 P\left(X,t\right)}{\partial X^2}$$

Now, we take the limit $\Delta x\to 0$, $\Delta t\to 0$ and assume that $Lim_{\substack{\Delta t\to 0\\ \Delta x\to 0}}\frac{\left(\Delta x\right)^2}{\Delta t}=1$. The rationale behind this assumption can be traced back to the parameters of the scaled random walk wherein we set the jump size as √(T/n) and time step as T/n. This choice of parameters clearly justifies the above limiting structure $Lim_{\substack{\Delta t\to 0\\ \Delta x\to 0}}\frac{\left(\Delta x\right)^2}{\Delta t}=1$ and we obtain the diffusion equation for the unbiased random walk as:

$$\frac{\partial P\left(X,t\right)}{\partial t}=\frac{1}{2}\frac{\partial^2 P\left(X,t\right)}{\partial X^2}$$