Financial Derivatives and Risk Management
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Lecture 38: Stochastic processes: Brownian Motion

n-step random walk

Let us, now, generalize to the n-step unscaled, unbiased random walk. For this purpose, we divide our time interval (0,T) into n+1 points 0, T/n, 2T/n,…, nT/n=T. The process is assumed to be located at the origin i.e. at (0,0) at t=0 and evolves in time by making a random jump of magnitude ±1 at each of the points T/n, 2T/n,…,T. Thus, the position of the process at t=T is given by its position at the immediately preceding evolution point i.e. (n-1)T/n or the (n-1)th step from which it will make the final jump at t=T to reach its final value. Since this final (nth) jump will also be of magnitude ±1 with probabilities ½, it can be represented by our binary random variable X_n where X_T takes the values ±1 with probabilities ½. We, thus, have the recursive relation for the n-step walk as:

$$W_n(T) = W_n\left(n\text{-}1.\frac{T}{n}\right) + X_n = W_n\left(n\text{-}2.\frac{T}{n}\right) + X_{n\text{-}1} + X_n = \sum_{i=1}^{n} X_i$$
$$E\left[W_n(T)\right] = E\left[\sum_{i=1}^{n} X_i\right] = \left[\sum_{i=1}^{n} E(X_i)\right] = 0$$
$$E\left[W_n(T)\right]^2 = E\left[\sum_{i=1}^{n} X_i\right]^2 = E\left[\sum_{i=1}^{n} X_i^2\right] = \left[\sum_{i=1}^{n} E\left(X_i^2\right)\right] = n$$
$$\sigma_T^2 = n$$

Thus, the position of the process at t=T can be expressed as the sum of the n binary random variables, each representing the evolution of the process at a time step. The mean value of the process remains zero while the variance of the process becomes equal to the number of time steps.

Thus, as we increase the number of steps, the variance increases with the number of steps. For a single step we had a variance of 1, for 2 steps we had a variance of 2 and when we have n steps we have a variance of n.

In fact, before we move forward, you can clearly see the Markov property here also. The jump at t=T is dependent entirely on the state of the process at time t=(n-1)T/n i.e. at the previous step and no earlier. So, the process's position at the immediately prior evolution point determines its evolution at the current step. The earlier history is irrelevant; it is simply the position at the immediately preceding step which will determine its position after the random step at the current evolution point. So, this is clearly a Markov process.

Scaled unbiased random walk

Thus, as the number of steps increases, the variance of the process blows up i.e. it diverges.

But our objective of generalizing the random walk is to evolve some mechanism of moving from the discrete time framework to the continuous time framework. Obviously, to achieve a continuous time environment, we need to squeeze the layer spacing i.e. the interval between two discrete time evolution points. Squeezing the time intervals implies increasing the number of time larger for larger the number of steps, smaller the interval between any two steps. The upshot is that we can achieve a continuous time flow by taking the limit n $\to\infty$.

However, as we increase the number of steps, the variance blows up and, indeed, in the limit n $\to\infty$, it diverges. Now, if the variance diverges, the process values diverge within a finite time after its evolution initiates. This makes the process unsuitable for modelling applications in finance and physics. So, we have to find a way out of this.

Now, we see that the variance is scaling as n where n is the number of steps. So, if we scale the jump size by 1/√n i.e. if we scale the binary random variable X'_i by $X_i/\sqrt{n}$, then we have:

$$E\left[W_n(T)\right]^2 = E\left[\sum_{i=1}^{n} X'_i\right]^2 = E\left[\sum_{i=1}^{n} X'^2_i\right] = \left[\sum_{i=1}^{n} E\left(X'^2_i\right)\right] = \left[\sum_{i=1}^{n} E\left(\frac{X_i}{\sqrt{n}}\right)^2\right] = \frac{1}{n}\left[\sum_{i=1}^{n} E\left(X_i^2\right)\right] = 1$$
$$\sigma_T^2 = 1$$

But that also makes no sense because now we end up with a mechanism where all processes have a uniform variance of 1 irrespective of the period of evolution. Thus, the variance, now, is independent of time of evolution. This is not observed in empirical behaviour where, as the time of evolution of the stochastic process increases we find that the variance also increases. Therefore, to have a uniform variance of 1 for all time evolution is really not resolving the problem.

We look at an alternative. The alternative scaling we consider is $Y_i = X_i\sqrt{(T/n)}$. Now, we have:

$$E\left[W_n(T)\right]^2 = E\left[\sum_{i=1}^{n} Y_i\right]^2 = E\left[\sum_{i=1}^{n} Y_i^2\right] = \left[\sum_{i=1}^{n} E\left(Y_i^2\right)\right] = \left[\sum_{i=1}^{n} E\left(X_i\sqrt{\frac{T}{n}}\right)^2\right] = \frac{T}{n}\left[\sum_{i=1}^{n} E\left(X_i^2\right)\right] = T$$
$$\sigma_T^2 = T$$

which turns out to be an acceptable value.

Hence, our complete set of assumptions is:

$(i)\ W_n(0) = 0$

(ii) *no of steps* $= n$ *so that, layer spacing* T/n,

(iii) *up and down jumps equal and of size* $\sqrt{\frac{T}{n}}$,

(iv) *up and down probabilities everywhere equal to* $\frac{1}{2}$.

Thus, the process starts at the origin. The number of steps is n and the layer spacing is T/n. The up and down jumps are √(T/n). The probability of up and down continues to be ½. So, we have an unbiased random walk. The recursive relation for the scaled walk i

$$W_n(T) = W_n\left(n-1.\frac{T}{n}\right) + Y_n = W_n\left(n-2.\frac{T}{n}\right) + Y_{n-1} + Y_n = \sum_{i=1}^{n} Y_i = \sqrt{\frac{T}{n}}\sum_{i=1}^{n} X_i$$ where

$$Y_i = \sqrt{\frac{T}{n}}\,X_i \text{ are independent identically distributed random variables}$$

$$Y_i = \begin{cases} +\sqrt{\frac{T}{n}} \text{ with } p\left(Y_i = +\sqrt{\frac{T}{n}}\right) = 1/2 \\ -\sqrt{\frac{T}{n}} \text{ with } p\left(Y_i = -\sqrt{\frac{T}{n}}\right) = 1/2 \end{cases}$$

Let us now, take some arbitrary point t=t* in (0,T) and work out the variance of the process at this time. The length of each time step is T/n. hence, the number of time steps to t* is: t*/(T/n)=nt*/T.

Hence,

$$W_n\left(t^*\right) = W_n\left(\frac{nt^*}{T}.\frac{T}{n}\right) = \sum_{i=1}^{nt^*/T} Y_i = \sqrt{\frac{T}{n}}\sum_{i=1}^{nt^*/T} X_i$$

$$E\left[W_n\left(t^*\right)\right] = 0; E\left[W_n\left(t^*\right)\right]^2 = \frac{T}{n}\left(\frac{nt^*}{T}\right) = t^* \text{ so that } Var\left[W_n\left(t^*\right)\right] = t^*$$

$W_n(t)$ is the spectrum of possible values that the process can take at time t. For example, $W_n(T/n)$ is the value that the process can take at the end of the first step i.e. ±√(T/n). Similarly, $W_n(t)$ is the spectrum of possible values that the process can take at time t i.e. at the end of nt/T steps.

This is illustration of the random walk. Gradually as we increase the number of steps, the zig zag also increases. As you can see here, the path that is being followed by random walk is continuous at all points. However, as the as the number of steps increases, the zig zagging would be so much that the paths would be nowhere differentiable, although they remain continuous.

Random walk to Brownian motion

Thus, we have now formulated a structure i.e. random walk which, albeit still being discrete, has n steps, the length of each spacing is T/n with the up and down jumps being of magnitude $\sqrt{(T/n)}$. In other words, at every evolution point of which there are n points, the process can jump by $\pm\sqrt{(T/n)}$, the probability of either the up-jump or the down-jump being ½.

At this point, we introduce the limit $n\to\infty$. This implies that we are squeezing the layer spacing T/n to infinitesimal values i.e. moving towards continuous time. Not only this, if $n\to\infty$, the step size $\sqrt{(T/n)}$ also approaches infinitesimally small values i.e. we also move into a continuous variable regime.

For resolving the continuous version, we can follow different approaches. We can invoke the Central Limit Theorem or the Diffusion PDE or the Stirling approximation.

Central Limit Theorem

Let us first state the theorem in the form relevant to our objective:

Let X_i; i=1,2,…,*n* be independent identically distributed random variables each with finite mean and variance μ and σ^2 respectively. Then the following expression is distributed as a standard normal variate.

$$Z_n = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}}$$

The cardinal feature of the Central Limit Theorem is that the limiting distribution invariably approaches the normal distribution irrespective of the underlying distributions of the random variables themselves. The only requirement is the finiteness of the means and variances of these underlying distributions.

The central limit theorem says if X_i, i=1,2,2,…,n are independent identically distributed random variables i.e. each has the same probability distribution and hence, the same mean and variance, then $Z_n = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}}$ in the limit that $n\to\infty$ approaches a standard normal distribution.

The special part of the CLT is that the underlying distribution of the various X_i are inconsequential. They maybe normally, binomially or uniformly distributed or whatever. In all cases, the limiting distribution approaches the normal distribution. All that is required is that the X_i should be independent, identically distributed and the mean and variance should be finite. That is the beauty of CLT that it does not care about the underlying distribution of the random variables.

Application of CLT to the Random Walk

Let us apply the CLT to $W_n(t)$ to obtain its distribution as $n \to \infty$. We have:

$$W_n(t) = W_n\left(\frac{nt}{T}.\frac{T}{n}\right) = \sum_{i=1}^{nt/T} Y_i = \sqrt{\frac{T}{n}} \sum_{i=1}^{nt/T} X_i$$

$$Also\ E(Y_i) = E\left(\sqrt{\frac{T}{n}} X_i\right) = \sqrt{\frac{T}{n}} E(X_i) = 0; E(Y_i^2) = E\left(\sqrt{\frac{T}{n}} X_i\right)^2 = \frac{T}{n} E(X_i^2) = \frac{T}{n}$$

$$E[W_n(t)] = 0; Var[W_n(t)] = t$$

$$In\ the\ case\ of\ the\ random\ walk$$

$$\mu = \mu_i = E(Y_i) = 0;\ E(Y_i^2) = \frac{T}{n}, \sigma^2 = \sigma_i^2 = \frac{T}{n}\ \forall\ i$$

$$Hence,\ by\ Central\ Limit\ Theorem\ \lim_{n\to\infty} \frac{\sum_{i=1}^{nt/T} Y_i - \frac{nt}{T}\mu}{\sqrt{\frac{nt}{T}\sigma^2}}$$

$$= \lim_{n\to\infty} \frac{\sum_{i=1}^{nt/T} Y_i}{\sqrt{\frac{nt}{T} \times \frac{T}{n}}} = \lim_{nt/T\to\infty} \frac{\sum_{i=1}^{nt/T} Y_i}{\sqrt{t}} \xrightarrow{distribution} N(0,1)$$

$$so\ that\ W(t) = W_\infty(t) = \lim_{nt/T\to\infty} \sum_{i=1}^{nt/T} Y_i \xrightarrow{distribution} N(0,t)$$

It may be noted that since we want to determine the distribution of $W_n(t)$ at time t which corresponds to nt/T number of time steps, which is therefore the number of random variables X_i i.e. the index shall extend from 1 to nt/T instead of 1 to n. So, the n factor in the CLT Z_n would be replaced by nt/T and the limit $n \to \infty$ will be replaced by $nt/T \to \infty$. Now, this does not affect the limit because if $n \to \infty$ then $nt/T \to \infty$ and vice versa as both t,T are finite.

In the limit that the number of time steps approaches infinity, the aforesaid construction of a scaled random walk converges to a mathematical structure called Brownian Motion W(t) that has certain well defined mathematical properties and plays a vital role in the modeling of stochastic processes.

BM is also sometimes called a Wiener Process

Properties of Brownian motion

The process $W = (W(t) : t \geq 0)$ *is a Brownian motion if and only if*

(i) CONTINUITY : $W(t)$ *is continuous, and* $W(0) = 0$,

(ii) DISTRIBUTION OF $W(t)$*: The value of* $W(t)$ *is distributed as a normal random* **var***iable* $N(0,t)$,

(iii) DISTRIBUTION OF INCREMENTS : The increment $W(s+t) - W(s)$ *is distributed as a normal* $N(0,t)$, *and is independent of the history of what the process did up to time s*.

(iv) *REPRESENTATION IN TERMS OF z*

we can express an increment of BM as $dW(t) = z\sqrt{dt}$

where z is $N(0,1)$ *distributed normal variate.*

(v) *DIFFERENTIABILITY*

The process $W(t)$ *is not differentiable at any point t*

(vi) *FRACTALITY*

BM is a self replicating object i.e. a FRACTAL.

BM plays a significant role in the modelling of stochastic processes. Just like we have straight line as the fundamental building block of deterministic curves, the Brownian motion structure constitutes the fundamental building block of stochastic processes. Let us retrace its properties:

The process W(t) initiates at the origin and thereafter evolves continuously with increasing time.

The possible values of the process, W(t), at any arbitrary instant of time t, are normally distributed with a mean of zero and a variance of t.

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The increment W(s+t)-W(s) of the process is also normally distributed with the mean of zero and a variance of (s+t)-s=t. Furthermore, BM is a Markov process i.e. this increment W(s+t)-W(s) is independent of the history of the process up to W(s). In other words, whatever has happened up to W(s) does not affect this increment. This increment W(s+t)-W(s) simply depends on where the process is at time s and thereafter it evolves randomly. How the system arrived at W(s) in irrelevant in determining the distribution of this increment.

Now, we can write the infinitesimal BM increment dW(t) in terms of standard normal variate also as z√(dt) where z is the standard normal variate. Clearly since dt is not random, z√(dt) is normally distributed with mean zero (since E(z)=0) and variance dt (since Var(z)=1).

The BM process is not differentiable due to the extensive zig zagging.

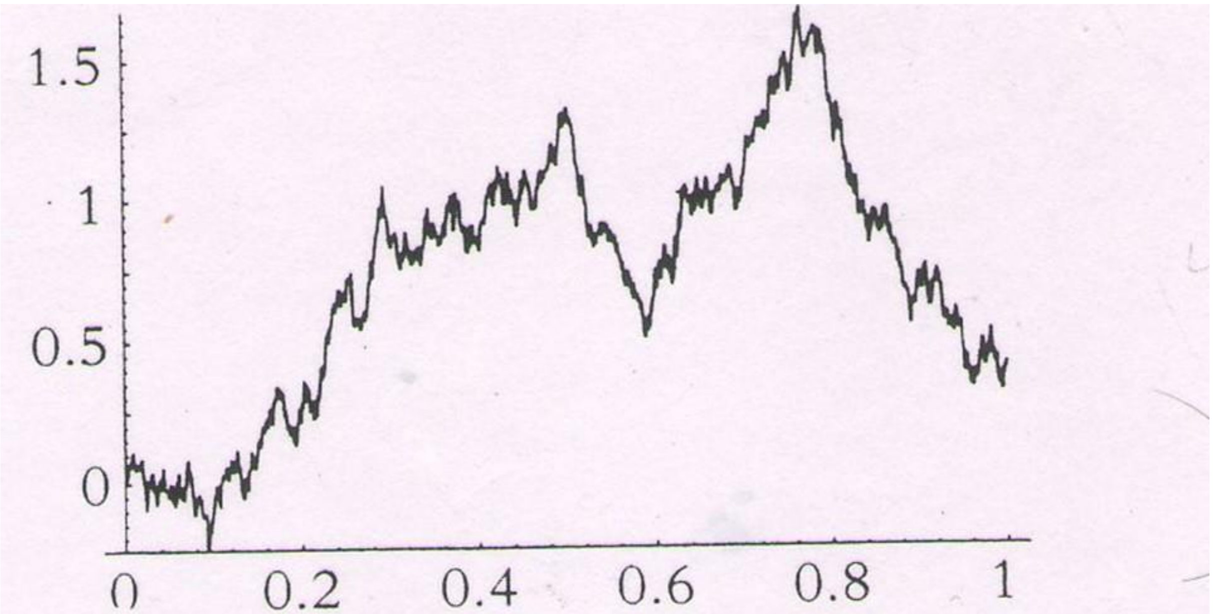
BM is a fractal. In other words, if we take any point on the Brownian motion, anew BM branch originates from there. In other words, the BM is self-replicating. If we look at it with a microscope with as much zoom as desired, it will still appear similar. That is what is a fractal.

Although W is continuous everywhere, it is (with probability one) differentiable nowhere.

Brownian motion will eventually hit any and every real value no matter how large, or how negative. It may be a million units above the axis, but it will (with probability one) be back down again to zero, by some later time.

Once Brownian motion hits a value, it hits it again infinitely often again from time to time in the future.

It doesn’t matter what scale you examine Brownian motion on – it looks just the same. Brownian motion is a fractal.



Now, the above is a particular realization of Brownian motion. This is one realization which has occurred. You start again you will get a different realization but this is a prototype of Brownian motion.

RANDOM WALKS	TIME LENGTH /LAYER SPACING	JUMP SIZE	EXP VALUE	VARIANCE
SINGLE STEP RW	T	±1	0	1
TWO STEP RW	T/2	±1	0	2
n-STEP RW	T/n	±1	0	n
n-STEP SCALED RW	T/n	$\pm\sqrt{(T/n)}$	0	T

n-STEP SCALED RW	**T/n**	**$\pm\sigma\sqrt{(T/n)}$**	**0**	**$\sigma^2 T$**
n-STEP SCALED RW WITH DRIFT	**T/n**	**$(\mu T/n)\pm \sigma\sqrt{(T/n)}$**	**μT**	**$\sigma^2 T$**
Brownian Motion	**->0**	**->0**	**0**	**T**
Scaled BM	**->0**	**->0**	**0**	**$\sigma^2 T$**

So, let us quickly summarize the results. For the single step random walk, the time length was T, the jump size was ±1, the mean was 0, the variance was 1. For a two-step walk, we had a mean of 0 and a variance of 2. For the n-step walk unscaled walk, the step time length became T/n because there were n steps and we retained the jump size as ±1 and we ended up with a variance of n. We then did scaling of the jump size, made the jump size vary as $\sqrt{(T/n)}$, the time length was retained at T/n and we ended up with a variance of capital T. Thus, now, we have the upswing and downswing as $\sqrt{(T/n)}$.

Now, this is the standard upswing and downswing. If we want to model a particular process that has vibration amplitude which is different from this standard, we can do that by introducing scaling factor in the jump size and use $\sigma\sqrt{(T/n)}$, where σ is called the scaling factor and we get the scaled BM. σ behaves as a scaling factor, it adjusts the amplitude of the fluctuations. In other words, it changes the dispersion of the process about the mean position. So, by changing the size of the jump size by adding σ, the variance also changes to $\sigma^2 T$.

In addition to this scaling of amplitudes, a drift term can also be added to reflect an upward or downward trend. In that case, the jump size would become $\mu T/n+\sigma\sqrt{(T/n)}$ where μ is the drift per unit time. In this case the mean μT and the variance continuous to be $\sigma^2 T$.

Thus, the infinitesimal increment of Brownian motion with drift is given by:

$dx=\mu dt+\sigma dW_t =\mu dt+\sigma z\sqrt{(dt)}$ where z is the standard normal variate.

Clearly, $E(dx)= \mu dt$ and $Var (dx)= \sigma^2 (dt)$

Example 1

A particle starts executing standard Brownian motion at t=0. What is the probability that the particle will be less than 10 units away from the origin after 100 units of time?

Solution

The spectrum of possible values of a BM W(t) at time t after initiation is normally distributed with a mean of 0 and a variance of t i.e. a standard deviation of $\sqrt{(t)}$.

Hence, the possible values after time t=100 are normally distributed with a mean of 0 and standard deviation of $\sqrt{(100)}$ =10.

We need to find the probability that $-10 < W(100) < +10$ i.e. $P(-10 < X < +10)$ where W(100)=X is N(0,100).

Expressing the above in terms of the standard normal variate $Z = \frac{X-\mu}{\sigma}$ we need to find out:

$$P\left(-10 < X < +10\right) = P\left(\frac{-10-0}{10} < Z < \frac{10-0}{10}\right) = P\left(-1 < Z < +1\right) = 0.6826$$

where the value has been obtained from normal tables.

Example 2

A particle executes scaled Brownian motion with drift in one dimension. The drift rate per unit time is 0.0001 units. The variance rate is 0.01. Calculate the probability that the particle is more than 2 units on the positive side of its initial position (origin) after 2500 units of time.

Solution

The spectrum of possible values of a BM with drift i.e.

$dx = \mu dt + \sigma dW_t = \mu dt + \sigma z\sqrt{(dt)}$

at time dt after initiation is normally distributed with a mean of μdt and a variance of $\sigma^2 dt$ i.e. a standard deviation of $\sigma\sqrt{(dt)}$.

Here μ=0.0001, σ^2=0.01 and t=2500.

Hence, the possible values after time t=2500 are normally distributed with a mean of 0.25 and standard deviation of $\sqrt{(0.01 * 2500)}$ =5.

Thus, after 2500 units of time, the mean position of the particle (as shown above) is 0.0001*2500=0.25 above the origin and the variance of the possible positions of the particle is 25 i.e. standard deviation is 5.

We need to find the probability of the particle being more than 2 units above the origin after 2500 units of time.

Thus, we need to find out $P(X>2.00)$ where X is $N(0.25,25)$

Converting to standard normal variate $P(X>2.00)=P[Z>(2.00-0.25)/5]=P(Z>0.35)=0.3632$

where the value has been obtained from normal tables.