Financial Derivatives & Risk Management
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Lecture 37 - Stochastic Processes: Random Walk

Hereon, the theory of the pricing of derivatives is introduced. However, these derivatives are mathematical functions of the prices of some underlying assets like stocks, bonds, currencies, indices or commodities. Hence, it becomes necessary to understand and develop the modelling of the prices of such underlying assets.

It is, obvious, that the future prices of such assets cannot be precisely predicted. They have a certain random component embedded in them. Indeed, were these prices to be fully predictable, the derivatives that we have talked about would have been absolutely useless and hence, worthless.

Types of evolutionary processes

On the basis of the behavior of processes or systems, in relation to the evolution with time, they can be classified into the following fundamental categories:

(i) Deterministic process
(ii) Stochastic processes
(iii) Chaotic processes

Deterministic processes are those processes whose future evolution is completely predictable. In other words, given the state of the system at one point in time, we can precisely ascertain its state at any futures point in time.

Clearly, in order that a system be deterministic:

(i) we must have a complete description of the initial state of the system i.e. we must have knowledge of all the variables that are required to facilitate a complete description of the system and we must have precise knowledge of their values at a given point in time. This would uniquely define the state vector of the system at that initial point.
(ii) We should have complete knowledge of the time evolution of the variables that completely describe the state vector, thereby enabling us to obtain the time evolved state vector at a future point in time through the time evolved values of the variables. The mathematical description of this time evolution usually takes the form of differential equations derived from some physical law that the defining variables are believed to obey.

So, two things are necessary, the initial state and the evolution equations which encapsulate the time evolution of the system. Once, we know these two, we can integrate them and we can obtain the state of the system at the future point in time. These are deterministic systems or deterministic processes.

Chaotic systems are a variant of these deterministic systems. They are essentially deterministic, but they have an additional characteristic. They are extremely sensitive to initial conditions. In other words, a small difference in the initial condition of two, otherwise identical, systems will manifest itself as an exponential blow up in the future evolution of the systems. Equivalently, a small error of measurement of initial conditions of such systems could result in the system evolving in an extremely different manner to that predicted on the basis of the original initial conditions. The chaotic map is extremely sensitive to initial conditions. It is sometimes said, in the context of chaotic systems, that a flap of a butterfly's wings in Bermuda could cause a cyclone in New York at the future point in time.

So any small difference in initial conditions manifests itself with exponential magnification as we move forward in time.

Stochastic systems are those systems whose evolution in time cannot be precisely predicted. In other words, they have a certain random component embedded in their evolution.

How that random component arises? It may arise due to

(i) incompleteness of the set of variables adopted for describing the state of the system;
(ii) imprecision of the measuring apparatus employed for measurement of variables describing the state of the system;
(iii) incomplete knowledge of the time evolution of the variables describing the system;
(iv) any combination of the above.

The outcome is that there is an element of unpredictability in the time evolution of the system. In other words, given a certain initial condition, given a certain value of the system's state at a point in time, we cannot precisely, exactly, estimate the value of the system's state at a future time.

Because the stochastic process evolves randomly in time, at every instant of time at which the process evolves, there is an element of randomness in the evolution of the process. Therefore, we can describe the evolution of the system by ascribing a random variable to each such instant of time at which the system evolves and as such, the entire process can be viewed as a collection, or a sequence of random variables indexed by time points at which the system makes a random transition. Thus, a stochastic process can be considered as a set $\left\{X_t ; t \geq 0\right\}$ where each X_t is a random variable defined on a suitable probability space.

And these random variables, can either be discrete or continuous. They will be indexed in terms of time points at which the system evolves or at which observations on the system are made, because we are talking about evolution in terms of time. Of course, at a very general level, we could have other indexing sets also, but for our purpose we shall always assume that this process is indexed in terms of time, because we are talking about evolution in terms of time.

Markov process

Now, there is a very important subset of stochastic processes, which are called Markov processes. A stochastic process is a general class of processes where the evolution is unpredictable. Markov process is a very specific, well defined class of stochastic processes, which has no memory. By no memory, we understand that the next step of the process, given its current position at a particular point in time, is independent of the manner i.e. the history or the path of how the process reached the current position. Its next step depends only on its current position. Given the position of the process at t=T, its location at t=T+1 depends only on its position at t=T and not on any earlier time e.g. t=T-1 etc.

Stated otherwise, the prior history of the process is irrelevant to the future evolution of the process. This is called a Markov process. The system or the process has no memory. Thus, a Markov process is a process, whose memory is restricted at any point of time to the immediately preceding time argument. So, whatever the state of the process is at t=T will determine how the process will evolve (of course, randomly) at t=T+1.

Obviously, the evolution would be random. But, it would not be affected at all by whatever happened in history prior to t=T. ***The basic thing is, the future random evolution of the Markovian system is dependent only on its present state and not conditioned upon the history of how the process reached the present state.*** In other words, the path followed up to the current time is irrelevant, it is only the state at time t, which is relevant.

Implications of Markovian property

Under the Markov assumption, all the joint probabilities are expressible as products of just two independent probabilities viz.

(i) The single-time probability $P_1(j,t_0)$

(ii) The two-time conditional probability $P_2(k,t|j,t_0)$

where $P_1(j,t_0)$ is the probability of the system being in state j at time t_0 and $P_2(k,t|j,t_0)$ being the two-time conditional probability of the system moving to state k at time t given that it was in state j at time t_0.

The evolution of any discrete stochastic process can be expressed in terms of the joint probability function $P_n(j_n,t_n;j_{n-1},t_{n-1};...;j_1,t_1)$ that represents the joint probability of the system being in state j_1 at time t_1, j_2 at time t_2,…, j_n at time t_n.

Let us look at this carefully. Obviously, any general stochastic process, not necessarily a Markov process, needs to be defined in terms of the probabilities of the states at which the system is going to be at various instants of time at which it evolves. A stochastic process that evolves starting from t_1 takes a random value in a sample space at t_2, another random value at t_3 and so on. In order to completely specify this process, we need to provide the joint probabilities of the system being in various states j_1 at t_1, j_2 at t_2 etc.

A stochastic process can be represented by a collection of random variables indexed by discrete tie points. Thus, to completely specify a stochastic process, we need to specify the entire path probabilities. In other words, the joint probabilities of all the random variables that constitute the process taking up various values at the points at which the process evolves.

To reiterate, in order to completely specify the system, we need the joint probability of the system, being in various states j_1, j_2, j_3 and so on at times t_1, t_2, t_3 … respectively. $P_n\left(j_n,t_n; j_{n-1},t_{n-1};...; j_1,t_1\right)$ is the n-time joint probability of the stochastic process taking a particular path between t_1 and t_n, the joint probability of various points on the path.

Now, applying Bayes theorem, $P(B;A)=P(B|A)P(A)$ and proceeding iteratively, we have,

$$P_n\left(j_n,t_n; j_{n-1},t_{n-1};...; j_1,t_1\right)=P_n\left(j_n,t_n\middle| j_{n-1},t_{n-1};...; j_1,t_1\right)\times$$
$$P_{n-1}\left(j_{n-1},t_{n-1}; j_{n-2},t_{n-2};...; j_1,t_1\right)$$
$$=P_n\left(j_n,t_n\middle| j_{n-1},t_{n-1};...; j_1,t_1\right)\times P_{n-1}\left(j_{n-1},t_{n-1}\middle| j_{n-2},t_{n-2};...; j_1,t_1\right)$$
$$\times P_{n-2}\left(j_{n-2},t_{n-2}; j_{n-3},t_{n-3};...; j_1,t_1\right)$$
$$=P_n\left(j_n,t_n\middle| j_{n-1},t_{n-1};...; j_1,t_1\right)\times P_{n-1}\left(j_{n-1},t_{n-1}\middle| j_{n-2},t_{n-2};...; j_1,t_1\right)$$
$$\times...\times P_2\left(j_2,t_2\middle| j_1,t_1\right)\times P_1\left(j_1,t_1\right)$$

This is simply splitting up the n-time joint probability into its various components in terms of conditional probabilities viz. n-time, (n-1)-time, …, 2-time conditional probabilities and the 1-time probability $P_n\left(j_n,t_n\middle| j_{n-1},t_{n-1};...; j_1,t_1\right)$, $P_{n-1}\left(j_{n-1},t_{n-1}\middle| j_{n-2},t_{n-2};...; j_1,t_1\right)$,…, $P_2\left(j_2,t_2\middle| j_1,t_1\right)$, $P_1\left(j_1,t_1\right)$.

Now, if the process is Markovian, we have

$$P_n\left(j_n,t_n\middle| j_{n-1},t_{n-1};...; j_1,t_1\right)=P_2\left(j_n,t_n\middle| j_{n-1},t_{n-1}\right)\ \forall n\geq 2$$
$$\textit{Therefore, for a Markov process}$$
$$P_n\left(j_n,t_n; j_{n-1},t_{n-1};...; j_1,t_1\right)=P_2\left(j_n,t_n\middle| j_{n-1},t_{n-1}\right)\times$$
$$P_2\left(j_{n-1},t_{n-1}\middle| j_{n-2},t_{n-2}\right)\times...\times P_2\left(j_2,t_2\middle| j_1,t_1\right)\times P_1\left(j_1,t_1\right)$$

Thus, the joint probability of a Markov process can be expressed in terms of various 2-time probabilities and the 1-time initial probability $P_1(j_1,t_1)$.

Now, $P_1(j_1,t_1)$ is the probability that the initial state i.e. the state of the system at $t=t_1$ is j_1. $P_2(j_2,t_2|j_1,t_1)$ is the conditional probability that the system in state j_1 at time t_1 evolves to the state j_2 at time t_2 and so on. So for a Markov process, we can write the joint probability of the entire stochastic process in terms of two-time probabilities $P_2(j_i,t_i|j_{i-1},t_{i-1})$ and the one-point probability $P_1(j_1,t_1)$.

Discrete time processes

Stochastic processes usually evolve with time. They are, therefore, indexed with reference to points on the timeline. In discrete time processes, time is assumed to evolve in discontinuous jumps i.e. in discrete steps of a certain length. Discrete time can be represented by a lattice with lattice points labelled by integers.

In discrete time processes, time is assumed to move in discrete chunks, in discrete steps. In other words, you cannot subdivide time indefinitely and arbitrarily. There is a lower bound up to which you can divide a period of time.

The fundamental property here is that the index set that is used for indexing the set of random variables that constitute the stochastic process is discrete i.e. consists of only discrete values. So, in discrete time process, time is assumed to evolve in discontinuous jumps e.g. $t=0,1,2,\ldots$

In such a scenario, you can represent time on a lattice where the points are regularly spaced and there is nothing in between. And the important thing is that the process when it evolves, it evolves only at the points on the lattice. It does not evolve in the interval between any two points on the lattice. It is only on the lattice points that the process makes a random transition from its immediately preceding value.

So, the discrete time process is a process where time is considered in discrete intervals. However, the unit of time may be of choice e.g. one day, one hour, one minute or whatever, depending on the nature of the process and the problem. The time unit is not an issue, but time steps when expressed in that unit must be discrete.

Thus, discrete time processes are those processes in which the system can change its state only at discrete instants of time. In other words, time may evolve continuously, the underlying time may evolve continuously but the system can change its state only at discrete points of time on the timeline.

Therefore, as far as the system's perspective is concerned, time is seen or time is apparently evolving in discontinuous jumps of a given length and that length constitutes the time step or the layer spacing. It is not evolving continuously because the system cannot change its state continuously, it can change its state only at given points in time, at discrete points in time.

Obviously we can represent the discrete time in terms of a lattice. The basic thing is, that in the case of a system which is evolving continuously we can still discretize the time on the basis of the points that which we are making the observation of the cardinal parameters which are defining the state of the system.

<u>Discrete variable process</u>

In discrete variable processes, the stochastic variable is assumed to evolve in discontinuous jumps i.e. in discrete steps of a certain length. In other words, we can always find two values of the variable such that no value of the variable lies between them.

Discrete variables can be represented by a lattice with lattice points labelled by integers.

Thus, the variable which is going to evolve can either evolve to take discrete values among a sample set or the sample set can be a continuous interval wherein the variable can take a particular but unpredictable value. Thus, the random variable can have a discrete spectrum or a continuous spectrum. If it has a discrete spectrum, it is called a discrete variable stochastic process. If it has a continuous spectrum, if it can take any value within a given interval, then it is said to be a continuous stochastic process. Just like time, if you have a discrete variable stochastic process, you can represent it by points on a lattice.

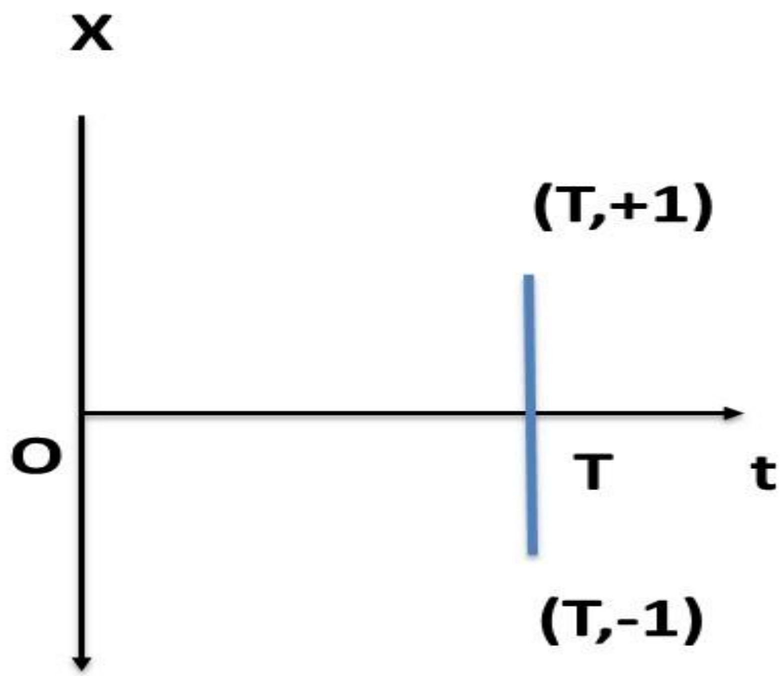
<u>One step unbiased unscaled random walk</u>

Consider a stochastic variable X_1 whose initial state (t=0) is represented by the origin (X=0 at t=0). Let time be modelled discretely with time step T so that the first time step is at t=T at which point the variable X_1 will make its first and only move. Let the move of the variable X_1 also be discrete of with step size 1 unit. Hence, at t=T, the variable X_1 will make a jump randomly i.e. unpredictably either to the value +1 or to the value -1 with the probability of each being ½. This is a single step unbiased unscaled random walk, single step because it involves only one-time step at which it makes a move, unbiased because the probability of the up-jump and down-jump are equal and unscaled because the jump size is independent of the number of steps. We write the position of the process at t=T as $W_1(T)$ where the subscript represents the number of jumps. Then,

$W_1(T)=X_1=\pm1$; $E[W_1(T)] = E(X_1) = ½ *-1 + ½ *+1 = 0$,
$E(X_1^2) = ½ *(-1)^2 + ½ *(+1)^2 = 1$ so that $\sigma_1^2=1$.

It may be noted here that the first and only time step is of length T, but it is only one step, so nothing can happen in between. The process starts at t=0, when the clock strikes t=T, it will make one jump, either up or down with equal chance of it being either. The size of the jump in either case is one unit.

Now, $W_1(T)$ is the set of possible positions of the process at time T i.e. at the end of one jump. Obviously $W_1(T)$ can take the values ±1 with probabilities ½ each. I can also write $W_1(T)$ as $W_1(T)=X_1$ where X_1 is a random variable that can take values ±1 with probabilities ½ each.



So, the critical parameters are the mean is 0 and the variance is 1. These are the important things. The variance is 1 and the mean is 0. This is the simplest possible random process. This is called the single step random walk.

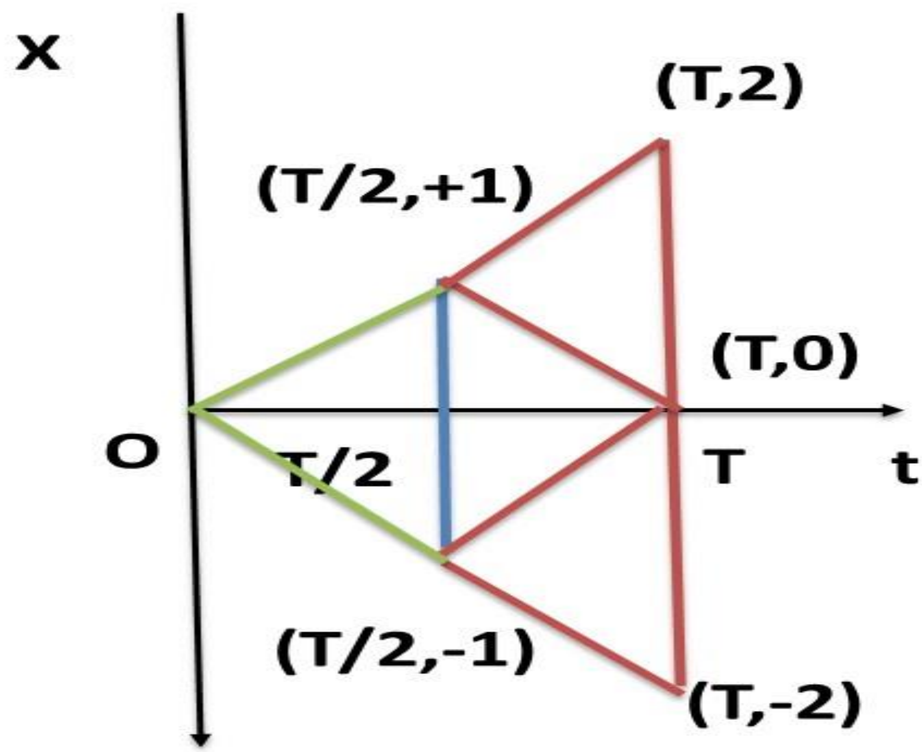
<u>Two step random walk</u>

Let us now increase the number of steps. Let us now look at a two-step random walk. However, we still retain the time interval (0,T). But instead of having a single jump at t=T, we assume that the process makes two jumps, first one at t=T/2 and the second one at t=T. Thus, it makes two jumps equally spaced in (0,T). Thus, the single step has now been replaced by two steps but within the same overall time span. The overall time is still T but we have now spilt (0,T) into two discrete intervals (0,T/2) and (T/2,T). But time is still discrete, we have two discrete points on the time-line, T/2 and T. The process does not do anything between (0,T/2), makes a jump at t=T/2, again remains inactive in (T/2,T) and makes a jump at t=T.

We represent the position of the process after the first jump at t=T/2 as $W_2(T/2)$. Then, clearly $W_2(T/2)=X_1$ where X_1 is defined as above as a binomial random variable that can take values ±1 with probability ½. This represents the first jump.

Now, the position of the process after the second step i.e. at time t=T i.e. $W_2(T)$ is given by the jump second jump (X_2, at t=T) from its position after the first jump so that $W_2(T)=W_2(T/2)+X_2=X_1+X_2$. We have,

$W_2(T)=W_2(T/2)+X_2=X_1+X_2=0,\pm 2$, $E[W_2(T)]=E(X_1+X_2)=0$,
$E[W_2(T)]^2=E(X_1+X_2)^2= E(X_1)^2+ E(X_2)^2+ 2E(X_1X_2) =2$, since $E(X_1X_2) =E(X_1) E(X_2) =0$ so that $\sigma_T^2=2$ as X_1 & X_2 are uncorrelated.



From the diagram, we see that the process will make its first jump at T/2. It will either be up or down to +1, or -1. Hence, at t=T/2, the process will be either at P(T/2,+1) or Q(T/2,-1). Now, we do not know whether it will be at P or Q, but it will be either at P or at Q. Now, from here it will make a second jump at t=T. If it is at P it makes a jump to either A(T,+2) or to B(T,0). These are the only two possibilities.

Similarly, if it is at Q(T/2,-1), then the up jump will bring it to B(T,0) and the down jump will bring it to C(T,-2). I emphasize once again, that this jump will take place at t=T. It will not take place earlier, time is discrete, the first jump takes place at t=T/2, the second jump takes place at t=T.

Clearly $W_2(T)$ can take the values -2,0,+2 with probabilities ¼. ½, ¼ respectively.

So, for the two-step binomial tree, the mean is zero, the mean continues to be zero, but the variance has changed from 1 to 2. This is important.

Appendix 1

Bayes Theorem

Suppose an event A can occur if and only if one of the hypotheses $B_1,\ldots,B_k$ is true. The probability $P(B_i)$ of occurrence of B_i is known for each i,i =1,...,k. Also known is the conditional probability $P(A/B_i)$ of occurrence of A given that B_i has already occurred, i=1,…,k. We want to find the conditional probability $P(B_i/A)$ of occurrence of B_i given that A has already occurred. This is given in Bayes theorem that follows.

To fix the idea suppose that a scientist observes a certain event A. He considers that A can happen only if one of the hypotheses $B_1,\ldots,B_k$ holds good. Before observing A, he assigns certain probabilities $P(B_1),\ldots,P(B_k)$ of these hypotheses to be true. He also knows the conditional probability $P(A/B_i)$ of occurrence of A when B_i is true. Now that he has observed the event A, how is he going to change his probabilities to different hypotheses $B_1,\ldots, B_k$? This is obtained by the conditional probability $P(B_i/A)$.

Generally, $P(B_i)$ and $P(B_i/A)$ will not be the same. Thus the occurrence of A generally changes one's assignment of probabilities to different hypotheses. The probabilities $P(B_i)$ which are assigned to B_i without any reference to A are called 'a priori' probabilities, i=1,…,k. The probabilities $P(B_i/A)$ which are calculated after A has been observed are called 'a posteriori' probabilities, i=1,…,k. Our main interest lies here in the hypotheses $B_1\ldots, B_k$.

Bayes theorem.

Let an event A occur only if one of the hypotheses $B_1,\ldots,B_k$ is true. Known are the probabilities $P(B_1),,\ldots, P(B_k)$ of occurrence of $B_1,\ldots, B_k$ respectively. The conditional probabilities $P(A/B_i)$, i=1,…,k, are also known. The posterior probability $P(B_i/A)$ is given by:

$$P\left(B_i \middle| A\right)=\frac{P\left(B_i\right)P\left(A \middle| B_i\right)}{\sum_{j=1}^{k} P\left(B_j\right)P\left(A \middle| B_j\right)}$$

provided at least one $P(B_t) > 0$, t=1,…,k.

Proof. We have $P(AB_i)= P(B_i)P(A/B_i)=P(A)P(B_i/A)$

Hence $P\left(B_i \middle| A\right)=\frac{P\left(B_i\right)P\left(A \middle| B_i\right)}{P(A)}=\frac{P\left(AB_i\right)}{P(A)}$. Now

$$P(A)=P\left(AB_1\right)+P\left(AB_2\right)+\cdots+P\left(AB_k\right)=\sum_{j=1}^{k} P\left(B_j\right)P\left(A \middle| B_j\right)=\sum_{j=1}^{k} P\left(AB_j\right)$$

Example 1

The manufacturing process employed by a company XYZ Ltd for its product can be carried out on two different machines viz. machine A and machine B. If the product is produced on machine A, the probability of it being defective is 0.05 and if it produced on machine B, the probability of being defective is 0.10. 60% of its production is carried out on machine A and 40% on machine B. The inspection department of the company during a regular inspection has come across one defective item. What is the probability that the defective item was processed on machine B?

Solution:

$$P\left(B \middle| Defective\right) = \frac{P\left(B \ \& \ Defective\right)}{P\left(Defective\right)}$$

$$P\left(Defective\right) = P\left(A \ \& \ Defective\right) + P\left(B \ \& \ Defective\right)$$

$$= 0.60 \times 0.05 + 0.40 \times 0.10 = 0.03 + 0.04 = 0.07$$

$$Hence, \ P\left(B \middle| Defective\right) = \frac{P\left(B \ \& \ Defective\right)}{P\left(Defective\right)} = \frac{0.04}{0.07} = 0.57$$

Example 2

The first of three urns contains 7 white and 10 black balls, the second contains 5 white and 12 black balls and the third contains 17 white balls and no black balls. A person chooses an urn at random and draws a ball from it. The ball is white. Find the probabilities that the ball came from (i) the first (ii) the second (iii) the third urn.

Solution

Let H_i by the hypothesis that the i^{th} urn was chosen and E be the event a white ball is drawn.

$$P\left(H_i\right) = \frac{1}{3}, \ i = 1, 2, 3, \ P\left(E \middle| H_1\right) = \frac{7}{17}, \ P\left(E \middle| H_2\right) = \frac{5}{17}, \ P\left(E \middle| H_3\right) = 1$$

$$P\left(H_1 \middle| E\right) = \frac{\frac{1}{3} \times \frac{7}{17}}{\frac{1}{3} \times \frac{7}{17} + \frac{1}{3} \times \frac{5}{17} + \frac{1}{3} \times 1} = \frac{7}{29},$$

$$P\left(H_2 \middle| E\right) = \frac{5}{29}, \ P\left(H_3 \middle| E\right) = \frac{17}{29}$$

Example 3

Two production lines manufacture the same type of items. In a given time line 1 turns out n_1 items of which n_1p_1 are defectives; in the same time, line 2 turns out n_2 items of which n_2p_2 are defectives. Suppose a unit is selected at random from the combined lot produced by the two lines.

Let D be the event of a defective item, A, the event the unit was produced by line 1 and B, the event it was produced by line 2. Determine P(A/D), P(B/D).

Solution

$$P(A) = \frac{n_1}{N},\ P(B) = \frac{n_2}{N}\ \ where\ \ N = n_1 + n_2$$

$$P(D|A) = p_1, P(D|B) = p_2$$

Hence $P(A|D) = \dfrac{P(A)P(D|A)}{P(A)P(D|A) + P(B)P(D|B)} = \dfrac{n_1 p_1}{n_1 p_1 + n_2 p_2}$

Similarly, $P(B|D) = \dfrac{n_2 p_2}{n_1 p_1 + n_2 p_2}$.

Appendix 2

Markov process

A Markov process is a stochastic process with the following properties:

(i) The number of possible outcomes or states is finite.
(ii) The outcome at any stage depends only on the outcome of the previous stage.
(iii) The probabilities are constant over time.

If x_0 is a vector which represents the initial state of a system, then there is a matrix M such that the state of the system after one iteration is given by the vector Mx_0. Thus we get a chain of state vectors: x_0, Mx_0, M^2x_0, . . . where the state of the system after n iterations is given by M^nx_0. Such a chain is called a Markov chain and the matrix M is called a transition matrix.

The transition matrix of an n-state Markov process is an n×n matrix M where the i, j entry of M represents the probability that an object is state j transitions into state i, that is if M = (m_{ij}) and the states are $S_1, S_2, \ldots, S_n$ then m_{ij} is the probability that an object in state S_j transitions to state S_i.

In other words, the transition matrix is given by $M=(m_{ij})$ where m_{ij} is the probability of configuration C_j making the transition to C_i.

Please note element m_{ij} is the transition probability from the j^{th} state to the i^{th} state.

The state vectors can be of one of two types: an absolute vector or a probability vector.

An absolute vector is a vector whose entries give the actual number of objects in a given state.

A probability vector is a vector where the entries give the percentage (or probability) of objects in a given state. Note that the entries of a probability vector add up to 1.

How to compute steady state of a Markov process

Let M be the transition matrix of a Markov process such that M^k has only positive entries for some k. Then there exists a unique probability vector $\mathbf{x}_s$ such that $M\mathbf{x}_s = \mathbf{x}_s$. $\mathbf{x}_s$ is called the steady state vector of the Markov process.

What remains is to determine the steady-state vector. Notice that we have the chain of equivalences:

$M\mathbf{x}_s = \mathbf{x}_s$, $M\mathbf{x}_s - \mathbf{x}_s = 0$, $M\mathbf{x}_s - I\mathbf{x}_s = 0$, $(M - I)\mathbf{x}_s = 0$, $\mathbf{x}_s$ is a vector in the null space of (M-I) i.e. $\mathbf{x}_s \in N(M - I)$

Thus $\mathbf{x}_s$ is a vector in the nullspace of M −I. If M^k has all positive entries for some k, then dim (N(M −I))=1 and any vector in N(M −I) is just a scalar multiple of $\mathbf{x}_s$. In particular, if $\mathbf{x} = (x_1,\ldots,x_n)^T$ is any non-zero vector in N(M − I), then $x_s = (1/c)\mathbf{x}$ where $c = x_1+x_2+\ldots+x_n$.

Example

A certain protein molecule can have three configurations which we denote as C_1, C_2 and C_3. Every second the protein molecule can make a transition from one configuration to another configuration with the following probabilities:

$$
\begin{array}{llll}
C_1 \rightarrow & C_2, P = 0.2 & C_1 \rightarrow & C_3, P = 0.5 \\
C_2 \rightarrow & C_1, P = 0.3 & C_2 \rightarrow & C_3, P = 0.2 \\
C_3 \rightarrow & C_1, P = 0.4 & C_3 \rightarrow & C_2, P = 0.2
\end{array}
$$

Find the transition matrix M and steady-state vector X_s for this Markov process.

Solution

The transition matrix for a Markov process is given by:

$M=(m_{ij})$ where m_{ij} is the probability of configuration C_j making the transition to C_i.

Therefore

$$
\begin{array}{c} \\ \\ \\ M = TO \end{array}
\begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} FROM \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.2 \\ 0.5 & 0.2 & 0.4 \end{pmatrix} \end{array}
\quad and \quad M - I = \begin{pmatrix} -0.7 & 0.3 & 0.4 \\ 0.2 & -0.5 & 0.2 \\ 0.5 & 0.2 & -0.6 \end{pmatrix}
$$

Now we compute a basis for N(M - I) by putting M - I into reduced echelon form:

$$
U = \begin{pmatrix} 1 & 0 & -0.8966 \\ 0 & 1 & -0.7586 \\ 0 & 0 & 0 \end{pmatrix} \quad and\ we\ see\ that\ X = \begin{pmatrix} 0.8966 \\ 0.7586 \\ 1 \end{pmatrix} \quad is\ the\ basis\ vector\ for\ N(M - I)
$$

Consequently, c=2.6552 and

$$
X_s = \begin{pmatrix} 0.3377 \\ 0.2857 \\ 0.3766 \end{pmatrix} \quad is\ the\ steady-state\ vector\ of\ this\ process.
$$

Alternatively, let x=(x,y,z) be the steady state vector. Then we have (M-I)**x**=0 so that

$$(M-I)\mathbf{x}=\begin{pmatrix}-0.7 & 0.3 & 0.4\\ 0.2 & -0.5 & 0.2\\ 0.5 & 0.2 & -0.6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0$$

$$or\ -0.7x+0.3y+0.4z=0 \qquad (1)$$

$$0.2x-0.5y+0.5z=0 \qquad (2)$$

$$0.5x+0.2y-0.6z=0 \qquad (3)$$

which can be solved togther with $x+y+z=1$ *to get the above result.*