Financial Derivatives and Risk Management
Professor J.P. Singh
Department of Management Studies
Indian Institute of Technology Roorkee
Lecture 33
American Options: Properties

Cost perspective vs cash flow perspective

When we work out the cash flows from an investment, we need to differentiate between cash outflows and cash inflows. In other words, the cash flows must be added algebraically. It is normally the practice to consider cash inflows as positive and cash outflows as negative although the sign reversal would not make any difference to the decision indicators provided they are interpreted consistently. Thus, a portfolio that generates a positive expected cash flow (expected cash inflow) at maturity would necessarily entail a negative cash flow (cash outflow) for its constitution.

Now, a cash outflow (negative cash flow) for the constitution of a portfolio can equivalently be considered as a positive cost for the portfolio construction. That is to say, cash outflows (considered negative in the cash flow perspective) incurred for the acquisition or the initiation of an investment may equally well be represented as positive costs for these activities. Viewed in this perspective (the cost perspective), a portfolio that generates a positive expected cash flow at maturity must necessarily entail a positive cost for its acquisition/constitution.

Stated summarily a cash outflow (negative) is equivalent to a positive cost and vice versa at portfolio inception. This is logical since the cost of acquisition represents the payment of the price by the investor and hence is a cash outflow. Obviously, the results obtained in portfolio theory remain unchanged by the use of either convention provided it is used consistently although the cash flow perspective makes more mathematical sense.

American puts: early exercise

Let us assume that at $t=0$ you have acquired an American put on a stock S with a strike price of K. Let the maturity of the put be $t=T$. Let $t=\tau$ be any arbitrary time point such that $0<\tau<T$. We need to examine the optimality of the exercise of the put at $t=\tau$. In particular, we need to establish the existence of at least one scenario such that it would be optimal to exercise the American put at $t=\tau$ i.e. before maturity T.

Recall that a put provides protection against a price fall i.e. if the price of the stock S falls below the exercise price K, the put option holder can still sell the asset at the exercise price K.

The first thing that is obvious at the outset is that earlier you exercise the put, sooner you receive the strike price K which can then be invested elsewhere to earn interest. So there is an upfront advantage of exercising the put as early as possible. Therefore, we need to examine what benefit would motivate the holder to defer the exercise of the put.

Now, the payoff from the exercise of a put at, say $t=\tau$ is given by $K-S_\tau$ if $S_\tau<K$. Obviously, this payoff increases as the stock price S_τ falls. The lower the stock price at exercise, the higher is the payoff from the put.

Consider a situation, say at time τ where $0<\tau<T$ at which the price of the stock is very low. In this situation, why would you hold on to the American put? Only if you expect the price of the underlying to go down further in the time (τ,T). But since the price is already very low, the chances of further fall are negligible. On the other hand, if you exercise the option at τ, you can invest the receipts elsewhere and earn interest income.

You would defer the exercise of the option, if you expect the stock price to fall further. If during (τ,T) you expect the price of the stock to go down further you will wait for your perception to materialize and the stock price to fall whence your payoff will increase. In such a situation you would defer the exercise of the put in anticipation of the price going down further.

Now, suppose S_τ is already very very low. It is at such a level that the probability of its going down any further is very very small. Then, obviously, there would be little rationale for deferring the exercise of the put. There would really be no logic, because you do not expect the price to fall any further. You are inevitably losing interest for every instant deferred and you are not likely (at least as per your perception) to benefit significantly by deferring the exercise of the put, because you feel the price is much too low right now to register any further fall.

And therefore the payoff at $t=\tau$ is likely to be the highest payoff that you could possibly get by put exercise.

Hence, in such a situation it would be optimal to exercise the put option early.

Therefore, we have established that there can be circumstances where early exercise of an American put may be optimal. There can be circumstances; there may be or there may not be but there can be situations where the early exercise of the put prior to maturity could be optimal.

And therefore, in the case of an American put, we must have $P\geq p$.

<u>American options: put-call parity</u>

Let us consider the following portfolio:

(i) Borrow an amount K at riskfree rate r for maturity T;
(ii) Write a European call costing c on the same stock S, with same exercise price K and maturity T;
(iii) Buy American put costing P on the stock S with exercise price K and maturity T;
(iv) Buy one unit of the stock S for S_0.

Now, because the portfolio contains an American put, the possibility of an early exercise of the American also needs to be considered. Hence, we need to analyse two cases here:

(i) When the American put is not exercised before maturity;
(ii) When the American put is exercised before maturity say at $t=\tau$ where $0<\tau<T$.

(i) **<u>When the American put is not exercised before maturity</u>**

Let us work out the payoff from this strategy at maturity of the options i.e. t=T. We have:

$\Pi(S_T)=-Ke^{rT}-\max(S_T-K,0)+\max(K-S_T,0)+S_T=-Ke^{rT}+\min(K-S_T,0)+\max(K-S_T,0)+S_T$
$=-Ke^{rT}+(K-S_T)+S_T<0$

Thus, the payoff of this strategy at maturity is negative. Further, the strategy does not involve any intermediate cash-flows during (0,T).

(ii) **<u>When the American put is exercised before maturity say at $t=\tau$ where $0<\tau<T$</u>**

Now, in this case, there are two important points viz (i) that at the time of exercise of the American put i.e. at $t=\tau$, the stock price must be lower than the strike price ($S_\tau<K$) because if the stock price exceeds the market price the option holder will be better by selling the stock in the market; and (ii) while the American put can be early exercised, the European call cannot be early exercised. Let the unexercised European call be worth c_τ at $t=\tau$. Let us work out the payoff from this strategy at $t=\tau$. We have, on using ($S_\tau<K$):

$\Pi(S_T)=-Ke^{r\tau}-c_\tau+\max(K-S_\tau,0)+S_\tau=-Ke^{r\tau}-c_\tau+K-S_\tau+S_\tau=-Ke^{r\tau}-c_\tau+K_\tau<0$ irrespective of the value of c_τ so long as $c_\tau\geq0$.

Thus, the payoff of this strategy at $t=\tau$ is also negative. Thus, in both cases (i) & (ii) i.e. irrespective of whether the American put is exercised at maturity or earlier, the cash flow at the point of exercise is invariably negative

Pertinent to point out here that if $S_\tau>K$, while the payoff on the American put will be 0, the market value of the European call will increase due to an increase in its intrinsic value (recall that the intrinsic value of a call at t is S_t-K). Because the portfolio contains a short call, the portfolio payoff will become more negative.

Hence, the no arbitrage requirement mandates that the cash out-flow at t=0 must be positive. But the cash outflow at t=0 is $K+c-P-S_0>0$. Hence, we must have:

$c+K>P+S_0$ or $c-P>S_0-K$

But, since early exercise of American calls on non-dividend stocks is always non-optimal, C=c, so that:

$C-P>S_0-K$

Now from the put-call parity for European options, we have:

$c + Ke^{-rT} = p + S_0$ or $c - p = S_0 - Ke^{-rT}$.

But $C = c$, $P \geq p$ so that $C - P \leq c - p$ whence $C - P \leq S_0 - Ke^{-rT}$. Thus, we have

$S_0 - K < C - P \leq S_0 - Ke^{-rT}$.

Alternatively, by transpositioning portfolio constituents:

	t=0	t=τ	t=T	
PORTFOLIO A		$S_\tau<K$	$S_T<K$	$S_T>K$
BUY EUROPEAN CALL	-c	0	0	S_T-K
INVEST	-K	$Ke^{(r\tau)}$	$Ke^{(rT)}$	$Ke^{(rT)}$
TOTAL	-(c+K)	$Ke^{(r\tau)}$	$Ke^{(rT)}$	$S_T+Ke^{(rT)}-K$

	t=0	t=τ	t=T	
PORTFOLIO A		$S_\tau<K$	$S_T<K$	$S_T>K$
TOTAL	-(c+K)	$Ke^{(r\tau)}$	$Ke^{(rT)}$	$S_T+Ke^{(rT)}-K$
PORTFOLIO B				
BUY STOCK	$-S_0$	S_τ	S_T	S_T
BUY AMERICAN PUT	-P	$K-S_\tau$	$K-S_T$	0
TOTAL	$-(S_0+P)$	K	K	S_T
$S_0+P < c+K$ but c=C, so $S_0-K < C-P$				

From previous slide: $S_0-K < C-P$
$c + Ke^{(-rT)}=p+S_0$
$c - p = S_0 - Ke^{(-rT)}$
c=C; p≤ P and c,p > 0
C-P ≤ c-p
$C - P \leq S_0 - Ke^{(-rT)}$
$S_0-K< C-P \leq S_0 - Ke^{(-rT)}$

So this is a very important relationship in respect of American calls and puts. You can see from here that in the case of European calls and puts we have an equality because both the options are exercisable at s ingle point in time.

But in the case of American options, because the options can be exercised over a time range, we cannot have an equality just a range within which the put, call premia must lie.

Impact of dividends on call prices

We, now, investigate the impact of a dividend announcement where the stock is to go ex-dividend during the life of the call. Let us say today is t=0 and the market receives the news of declaration of dividend on the stock underlying an option with the ex-dividend date being within the tenure of the option (0,T).. What is the effect of this dividend declaration on the call price at t=0 when it is made? That is the question proposed to be addressed.

Let us try to understand the dynamics. Even after dividend is declared by the company, the trades in the stock continue in the normal course. The important point is that a date is stipulated by the company such that dividend is actually distributed to all the persons whose names appear in the Register of Members as on that specified date. This date specified by the company is called the Record Date. On the basis of this record date indicated by the company another date usually one or two days prior to the record date called the ex-dividend date is notified by the exchanges at which the stock is listed for trading. As long as the stock is purchased prior to the ex-dividend date, the buyer has the right to receive the dividend (and therefore the price that he pays is called the cum-dividend price). In fact, this buyer can also sell the stock any time on or after the ex-dividend date and still receive the dividend. However, when the stock is sold after the ex-dividend date, the buyer is not entitled to the dividend which goes to the seller if he has acquired the stock before the ex-dividend date. Therefore, this price is now called the ex-dividend price.

Now, because the right to dividend shifts from the buyer to the seller on the ex-dividend date, the price of the stock traded before the ex-dividend date (cum dividend price) exceeds the price after the ex-dividend date (ex-dividend price) by the amount of dividend. Equivalently, the stock price registers a discontinuous fall equal to the amount of dividend on going ex-dividend.

Now the payoff of a call is $\max(S_T\text{-}K,0)$. Clearly a fall in the stock price manifests as a reduction in the payoff. Now, if the stock is going ex-dividend within the tenure of the option (0,T), the stock price is going to fall by the amount of dividend on the ex-dividend date and this ex-dividend date is within the maturity of the option. Naturally, the expected payoff on the option will also decline compared to the status of no-dividend. Accordingly, the call price registers a fall on the dividend announcement provided the stock goes ex-dividend during the call's tenure.

The converse is the case in the case of put options.

Put-call parity of American options with dividends

As shown in the case without dividends, when there are no dividends

$C - P \leq S_0 - K_e^{-rT}$

Now, dividends reduce C and increase P. Hence relationship must also be true when there are dividends.

Let the underlying deliver dividends of D_τ at $t=\tau$ with $0<\tau<T$ i.e. during the lifetime of the options. We retain the same arbitrage portfolio as in the previous case with the modification that we borrow $K+D_0$ instead of K only and obtain:

(i) Borrow an amount $K+D_0$ at riskfree rate r for maturity T;
(ii) Write a European call costing c on the same stock S, with same exercise price K and maturity T;
(iii) Buy American put costing P on the stock S with exercise price K and maturity T;
(iv) Buy one unit of the stock S for S_0.

Now, because the portfolio contains an American put, the possibility of an early exercise of the American also needs to be considered. Hence, we need to analyse two cases here:

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(i) **When the American put is not exercised before maturity**

Let us work out the payoff from this strategy at maturity of the options i.e. t=T. We have:

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$=-Ke^{rT}-D_T+\min(K-S_T,0)+\max(K-S_T,0)+S_T+D_T$
$=-Ke^{rT}-D_T+(K-S_T)+ S_T+D_T<0$

Thus, the payoff of this strategy at maturity is negative. Further, the strategy does not involve any intermediate cash-flows during (0,T).

(ii) **When the American put is exercised before maturity say at $t= \tau$ where $0<\tau<T$**

Now, in this case, there are two important points viz (i) that at the time of exercise of the American put i.e. at $t=\tau$, the stock price must be lower than the strike price ($S_\tau<K$) because if the stock price exceeds the market price the option holder will be better by selling the stock in the market; and (ii) while the American put can be early exercised, the European call cannot be early exercised. Let the unexercised European call be worth c_τ at $t=\tau$. Let us work out the payoff from this strategy at $t=\tau$. (a) If the dividend has been received on the stock before $t=\tau$, then the payoff from the portfolio is, on using ($S_\tau<K$):

$\Pi(S_\tau)=-Ke^{r\tau}-D_\tau-c_\tau+\max(K-S_\tau,0)+S_\tau+D_\tau$
$=-Ke^{r\tau}- D_\tau - c_\tau +K- S_\tau+S_\tau+D_\tau =-Ke^{r\tau}-c_\tau +K_\tau<0$ irrespective of the value of c_τ so long as $c_\tau\geq 0$.

Thus, the payoff of this strategy at $t=\tau$ is also negative.

(b) If the dividend is received after exercise i.e. after $t=\tau$, then the payoff at $t=\tau$ will be:

$\Pi(S_\tau)=-Ke^{r\tau}-D_\tau-c_\tau+\max(K-S_\tau,0)+S_\tau$
$=-Ke^{r\tau}- D_\tau - c_\tau +K- S_\tau+S_\tau =-Ke^{r\tau}- D_\tau-c_\tau +K_\tau<0$ irrespective of the value of c_τ so long as $c_\tau\geq 0$.

Thus, in both cases (i) & (iia), (iib) i.e. irrespective of whether the American put is exercised at maturity or earlier, the cash flow at the point of exercise is invariably negative

Pertinent to point out here that if $S_\tau>K$, while the payoff on the American put will be 0, the market value of the European call will increase due to an increase in its intrinsic value (recall that the intrinsic value of a call at t is S_t-K). Because the portfolio contains a short call, the portfolio payoff will become more negative.

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But, since early exercise of American calls on non-dividend stocks is always non-optimal, C=c, so that:

$C-P>S_0-D_0-K$

Thus, $S_0-D_0-K < C - P \leq S_0 - K_e^{-rT}$