Financial Derivatives and Risk Management
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Lecture 32: Options: Price bound, American Options

Put-call parity with yield on underlying asset

Now, we consider the case where we have a percentage yield, on the underlying asset i.e. the carrying of the underlying asset provides the investor with a percentage yield on it. Let us assume that the yield rate is q on continuous compounding basis. Clearly, then an investment in one unit of the underlying asset at t=0 will result in it growing to $\exp^{(qT)}$ units at time t=T where t=T is, as usual, the maturity of the options. Stated equivalently, to get one unit of the underlying asset at t=T, we need to possess or acquire its $e^{(-qT)}$ units at t=0 which will obviously entail an investment cost of $S_0e^{(-qT)}$. Hence, we consider the following portfolio:

(i) Borrow an amount Ke^{-rT} @ riskfree rate r for maturity T;
(ii) Write a European call on the same stock S, with same exercise price K and maturity T;
(iii) Buy European put on the stock S with exercise price K and maturity T;
(iv) Buy e^{-qT} unit of the stock S for S_0e^{-qT}.

Let us work out the payoff from this strategy at maturity of the options i.e. t=T. We have:

$\Pi(S_T)=-K-\max(S_T-K,0)+\max(K-S_T,0)+S_T=-K+\min(K-S_T,0)+\max(K-S_T,0)+S_T=-K+(K-S_T)+S_T=0$

Thus, the payoff of this strategy at maturity is 0. Further, the strategy does not involve any intermediate cash-flows during (0,T). Hence, the no arbitrage requirement mandates that the cash out-flow at t=0 must also be 0. But the cash outflow at t=0 is $+Ke^{-rT}+c-p-S_0e^{-qT}$. Hence, we must have:

$c+Ke^{-rT}=p+S_0e^{-qT}$

It may be noted that the default risk on the option contracts is ignored and these contracts are assumed default-free. Since the stock position is assumed long i.e. the stock is assumed in possession of the investor, its sale proceeds are assured to be received. The repayment of borrowings of Ke^{-rT} are to be met from the proceeds of the other components of the portfolio (which are whose realizations are riskfree). Hence, the repayment of borrowings is also riskfree which justifies the use of the riskfree rate. Thus, the entire procedure is devoid of risk under the given assumptions. It follows from the no arbitrage requirements that they must be priced equally at equilibrium.

Remember the relation assumes:

(i) Both options are European style.
(ii) Both options have the same underlying S.
(iii) Both options have the same exercise price K.
(iv) Both options have the same maturity T.
(v) Option contracts are default free.

In the previous case (Dollar income) the stock generated a certain money amount, the dividend was given in terms of a money amount. In the present case, the income on the stock is given as a percentage return (yield) in continuous compounding terms. If the investor holds the underlying asset he gets a certain percentage yield on the value of the underlying asset.

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
BUY CALL	-c	0	S_T-K
INVEST	$-Ke^{(-rT)}$	K	K
TOTAL	$-c-Ke^{(-rT)}$	K	S_T

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
TOTAL	$-c-Ke^{(-rT)}$	K	S_T
PORTFOLIO B			
BUY ASSET	$-S_0e^{(-qT)}$	S_T	S_T
BUY PUT	-p	$K-S_T$	0
TOTAL	$-S_0e^{(-qT)}-p$	K	S_T
$c+Ke^{(-rT)} = S_0e^{(-qT)}+p$			

Now, in the previous cases we had bought one unit of the stock in the arbitrage portfolio whereas in the present case we long only e^{-qT} units of the stock at t=0. This is because the stock value is growing at a continuously compounded rate of q. Therefore, in order to have one unit of the stock at option maturity, we need only e^{-qT} units at t=0. This will grow to $(e^{-qT})^{qT}=1$ unit at t=T which will be used for arbitrage if the no arbitrage balance is disturbed.

This (e^{-qT}) will cost $S_0(e^{-qT})$ of investment at t=0. This investment will realize one unit of the stock S which when sold in the market at t=T will fetch S_T.

<u>Bounds on option prices</u>

It is emphasized at the outset that these bounds are arrived at by no-arbitrage considerations. In other words, if these bounds are violated then their arbitrage will set in to rectify the disequilibrium. It is not that these bounds cannot be violated at all. There may be situations in the market when these bounds are violated but if these bounds are violated, arbitrage pressure will force the prices to realign themselves in such a way that the equilibrium prices move back to within the bounds.

Bounds on European calls

Lower bound on European calls

We form the following arbitrage portfolio:

Hence, we consider the following portfolio:

(i) Borrow an amount Ke^{-rT} @ riskfree rate r for maturity T;
(ii) Write a European call on the same stock S, with same exercise price K and maturity T;
(iii) Buy one unit of the stock S for S_0.

Let us work out the payoff from this strategy at maturity of the options i.e. t=T. We have:

$\Pi(S_T)=-K-\max(S_T-K,0)+S_T=-K+\min(K-S_T,0)+S_T=-(K-S_T)+\min(K-S_T,0)\leq 0$

Thus, the payoff of this strategy at maturity is non-positive. Further, the strategy does not involve any intermediate cash-flows during (0,T). Hence, the no arbitrage requirement mandates that the cash out-flow at t=0 must be non-negative. But the cash outflow at t=0 is $+Ke^{-rT}+c-S_0$. Hence, we must have:

$c+Ke^{-rT}-S_0\geq 0$ or $c\geq S_0-Ke^{-rT}$

which gives us the lower bound on the price of a European call at a given point in time.

	t=0	t=T	
		$S_T<K$	$S_T>K$
SELL CALL	c	0	$-(S_T-K)$
BUY STOCK	$-S_0$	S_T	S_T
BORROW	$Ke^{(-rT)}$	-K	-K
TOTAL	$-c-Ke^{(-rT)}+S_0$	$S_T-K<0$	0
Cash flow at t=0: $-S_0+c+Ke^{(-rT)}>0$ or $c>S_0-Ke^{(-rT)}$			

There is no state at maturity (t=T) in which the cash flow turns out to be positive. This means that an investor investing in this portfolio either you are going to get a 0 cashflow or a negative cash flow at maturity. There is no probability of a positive cashflow on maturity.

Because there is some finite probability, howsoever small, that the state with negative cashflow can occur, this portfolio will command a negative price at t=0.

In other words, this portfolio is worse than a portfolio which gives zero cashflows under all circumstances. Suppose there is a portfolio W that gives 0 cashflows under all possible states of nature at t=T. Clearly, this portfolio W will command zero price at t=0. Then, the given arbitrage portfolio is worse than W because it gives a negative cashflow under one possible state of nature and 0 in the only other possible state, while W gives 0 in both the states.

Therefore, the arbitrage portfolio will carry a price lower than the price of W i.e. a negative price because the price of W is zero.

Upper Bound on European calls

Construct the following arbitrage portfolio:

(i) Write a European call on the same stock S, with same exercise price K and maturity T;
(iii) Buy one unit of the stock S for S_0.

Let us work out the payoff from this strategy at maturity of the options i.e. t=T. We have:

$\Pi(S_T)=-\max(S_T-K,0)+S_T=\min(K-S_T,0)+S_T=\min(K,S_T)\geq 0$

Thus, the payoff of this strategy at maturity is non-negative. Further, the strategy does not involve any intermediate cash-flows during (0,T). Hence, the no arbitrage requirement mandates that the cash out-flow at t=0 must be non-positive. But the cash outflow at t=0 is $c-S_0$. Hence, we must have:

$c-S_0\leq 0$ or $c\leq S_0$

which gives us the upper bound on the price of a European call at a given point in time.

The explanation of this upper bound is simple. The call gives the holder the right to buy the stock at a particular pre-determined price. Therefore, the price of the call can never exceed the price of that underlying stock at that instant of time The best that the call holder can do with the call is buy the stock and so the value of the right embedded in the call can never exceed than the value of the stock.

Lower bound on European puts

We form the following arbitrage portfolio:

(i) Borrow an amount Ke^{-rT} @ riskfree rate r for maturity T;
(ii) Buy a European put on the same stock S, with same exercise price K and maturity T;
(iii) Buy one unit of the stock S for S_0.

Let us work out the payoff from this strategy at maturity of the options i.e. t=T. We have:

$\Pi(S_T)=-K+\max(K-S_T,0)+S_T=-K+\max(K,S_T)-S_T+S_T=-K+\max(K,S_T)\geq 0$

Thus, the payoff of this strategy at maturity is non-negative. Further, the strategy does not involve any intermediate cash-flows during (0,T). Hence, the no arbitrage requirement mandates that the cashflow at t=0 must be non-positive. But the cash outflow at t=0 is $+Ke^{-rT}-p-S_0$. Hence, we must have:

$Ke^{-rT}-p-S_0\leq 0$ or $p\geq Ke^{-rT}-S_0$

which gives us the lower bound on the price of a European put at a given point in time.

	t=0	t=T	
		$S_T<K$	$S_T>K$
BUY PUT	-p	$K-S_T$	0
BUY STOCK	$-S_0$	S_T	S_T
BORROW	$Ke^{(-rT)}$	-K	-K
TOTAL	$Ke^{(-rT)}-S_0-p>0$	0	$S_T-K>0$
Cashflow at t=0: $Ke^{(-rT)}-S_0-p<0$ or $p>Ke^{(-rT)}-S_0$			

Upper bound on European put

(i) Borrow an amount Ke^{-rT} @ riskfree rate r for maturity T;
(ii) Buy a European put on the same stock S, with same exercise price K and maturity T.
Let us work out the payoff from this strategy at maturity of the options i.e. t=T. We have:

$\Pi(S_T)=-K+\max(K-S_T,0)=-K+\max(K,S_T)-S_T\leq 0$

Thus, the payoff of this strategy at maturity is non-positive. Further, the strategy does not involve any intermediate cash-flows during (0,T). Hence, the no arbitrage requirement mandates that the cash out-flow at t=0 must be non-negative. But the cash outflow at t=0 is $Ke^{-rT}-p$. Hence, we must have:

$Ke^{-rT}-p\geq 0$ or $p\leq Ke^{-rT}$

which gives us the upper bound on the price of a European put at a given point in time.

A put gives the holder a right to sell the stock at a particular pre-determined price. Therefore, its value can never exceed that predetermined price. However, that pre-determined price will be receivable at the maturity of the option. Hence, the value of the put at any earlier instant will not exceed the present value of that pre-determined price (exercise price) worked out at that earlier instant.

Bounds on European options on dividend paying stocks

	t=0	t=T	
		$S_T<K$	$S_T>K$
BUY CALL	-c	0	S_T-K
SHORT STOCK	$+S_0$	$-S_T-D_T$	$-S_T-D_T$

INVEST	**$-Ke^{(-rT)}$ $-D_0$**	**$K+D_T$**	**$K+D_T$**
TOTAL	**$-c-Ke^{(-rT)}+S_0-D_0$**	**$K-S_T > 0$**	**0**
Cashflow at t=0: $-(c+Ke^{(-rT)}+D_0)-S_0<0$ or $c > S_0-D_0-Ke^{(-rT)}$			

AMERICAN OPTIONS

American options can be exercised at any time up to the expiration date, whereas European options can be exercised only on the expiration date itself. Since, this adds flexibility to the option contract by broadening the domain of exercise of the option, it must be that:

$C \geq c$ $\qquad$ $P \geq p$

i.e. the value of the American option shall be, at least to its European counterpart.

Arbitrage comes into play when we argue that if the payoff of a portfolio is positive (negative) at maturity then the cashflow for creating the portfolio at t=0 must necessarily be negative (positive). A positive payoff must command a positive price (negative cashflow for setting up the portfolio) and vice versa.

Early exercise of American calls on non-dividend paying stocks

Let us assume that you have taken a long position in an American call at t=0 on an underlying stock S with a strike price K. Let the maturity of the call be at t=T, although being an American option it can be exercised at any time in (0,T). let the stock S be expected to NOT pay any dividend during the interval (0,T).

We examine the optimal of the exercise of this call at a time point τ such that $0<\tau<T$.

The first thing we observe is that it must be that $S_\tau>K$, because if it not so, the payoff from the exercise of the call would be 0 and hence, the exercise would make no sense at all.

Hence, we conclude that if at all, the call is to be exercised at $t=\tau$, then $S_\tau>K$.

Now, since this is a call option, it gives the holder the right to buy the stock S at the pre-determined strike price K. Thus, if you exercise the option at $t=\tau$, you will pay the strike price K and receive one unit of stock S that has a market price of S_τ at that instant.

Now, there can be two scenarios:

(A) You hold the stock with you till the option maturity t=T; or
(B) You sell the stock at any time say t^* in the interval $\tau<t^*<T$ i.e. before the maturity of the option.

Case (A): Holding the stock till maturity of option

In this case, on the maturity date of the option i.e. t=T, your holdings shall consist of one unit of stock S and zero options since you have already exercised the option.

You can achieve the same holdings on this date t=T by exercising the option at t=T itself. However, this strategy has two clear advantages viz.

(i) You pay the same predetermined price K on the maturity date i.e. t=T, and not at the earlier date $t=\tau$ (in case you had early-exercised the call). Therefore, you save the interest cost for the period $t=\tau$ to t=T on the predetermined price K while achieving the same portfolio at t=T.

(ii) You continue to enjoy insurance against a fall in the stock price S below the exercise price K upto t=T i.e. if the stock price S happens to fall below the exercise price K at t=T, you still have the discretion not to exercise the option and either retract the decision to invest in that particular stock S or even buy the stock S in the market at a lower price if you so desire. In either case, the loss will simply be confined to the amount of premium paid for acquiring the option.

But, if you have already exercised the option at $t=\tau$, then you own the stock and therefore, all the loss on account of the fall in stock price will necessarily be borne by you. Option premium is also to be borne by you since you have already bought the option.

If you had early exercised the call at $t=\tau$, then you had acquired the stock on exercising the call at $t=\tau$ and paying the exercise price K at that instant. Assuming that you had borrowed this amount at the riskfree rate, the amount of repayment to be made by you at option maturity i.e. t=T is $Ke^{r(T-\tau)}$. Thus, the worth of your portfolio at t=T would be $S_T-Ke^{r(T-\tau)}$.

If you exercise the call at maturity t=T, the intrinsic value of the American call at t=T is S_T-K. This represents the immediate payoff from the option if it is exercised at t=T (i.e. if you or the unexercised call buyer exercises the option at t=T) because the exercise of the option at t=T will give the exerciser a unit of stock with current market worth S_T at an exercise price K. It follows that the value of the unexercised call at t=T cannot fall below S_T-K. Thus, the value of the portfolio consisting of the unexercised call would be S_T-K. Also, if you exercise the call you still end up with a portfolio value of S_T-K.

It, therefore, follows it is advantageous to retain the call unexercised until maturity and exercise it at maturity if it is desired to acquire the stock or even sell it unexercised rather than exercising it early and then selling the stock at maturity.

Case (B): Selling the underlying stock before maturity of option e.g. at t^* ($\tau<t^*<T$)

In this case, we compare the two scenarios:

(i) Sell the stock that is received on exercising the option at $t=\tau$; or
(ii) Sell the option without exercising.

In case (i), you had acquired the stock on exercising the call at $t=\tau$ and paying the exercise price K at that instant. Let us assume that the you sell the stock (received by you on call exercise at $t=\tau$) at t^* where ($\tau<t^*<T$) for S_{t^*}. Assuming that you had borrowed the exercise price K at the riskfree rate, the amount of repayment to be made by you at the time of selling the stock

i.e. at $t=t^*$ will be $Ke^{r(t^*-\tau)}$. Hence, the profit on the deal i.e. exercising the option at $t=\tau$, getting the stock for K at that instant and then selling it at $t=t^*$ will be $S_{t^*}-Ke^{r(t^*-\tau)}$. Thus, the worth of your portfolio at $t=t^*$ would be $S_{t^*}-Ke^{r(t^*-\tau)}$(in cash).

In the latter strategy i.e. case (ii), the intrinsic value of the American call at $t=t^*$ is S_{t^*}-K. This represents the immediate payoff from the option if it is exercised at $t=t^*$ (i.e. if the unexercised call buyer exercises the option at $t=t^*$) because the exercise of the option at $t=t^*$ will give the exerciser a unit of stock with current market worth S_{t^*} at an exercise price K. It follows that the value of the unexercised call at $t=t^*$ cannot fall below S_{t^*}-K. Thus, if the unexercised call is sold by you at $t=t^*$, the value of the portfolio would be at least S_{t^*}-K (in cash).

It, therefore, follows that even in case (B), it is advantageous to sell the call unexercised at any instant in the interval $0<t^*<T$ rather than first exercising it, acquiring the stock and then selling the stock.

The important thing here is that in the latter strategy, the buyer of the unexercised option at $t=t^*$ can do everything embedded in the former strategy by simply exercising the option. Besides, buyer enjoys additional benefit of insurance against a price fall of the underlying upto the maturity of the call. He can decide to retract his decision to invest or buy the stock from the market if the prices evolve in such a way.

The buyer will, therefore, prefer and pay more for (ii) than for (i). He enjoys additional benefit of insurance against a price fall of the underlying if he wants to hold the option till maturity.

Furthermore, the option will be bought by another investor who does want to hold the stock. Such investors must exist, otherwise the current stock price would not be sustained. The price obtained for the option will, in fact, be greater than its intrinsic value of S_{t^*}-K.

Thus, we have shown that it is never optimal to early exercise the American call. In other words, the investor does not acquire any additional benefit due to the flexibility provided by the time of exercise of American call. This means that there is no benefit attached to the American call because it will never be optimal to exercise the American call earlier than maturity. But European calls are exercisable on maturity, so American calls, to that extent, do not entitle the investor any benefit due to flexibility of exercise on account of non-optimality.

It follows as a corollary that the price of an American call on a non-dividend paying stock shall be equal to that of a European call and not exceed it i.e. C=c.

<u>Quantitative argument</u>

We know that the lower bound on the price of a European call at an arbitrary $t=\tau$ in (0,T) $c \geq S_\tau - Ke^{-r(T-\tau)}$ where T-τ represents the remaining term to maturity, T being the original maturity and τ being the time elapsed till now.

Further, because the American option carries the additional flexibility of time of exercise, it would cost either as much or more than the European counterpart so that C≥c.

Hence, we must have $C \geq S_\tau - Ke^{-r(T-\tau)}$. But since r>0, $K > Ke^{-r(T-\tau)}$ so that $C > S_\tau - K$.

But, S_τ-K is simply the intrinsic value of the call i.e. the profit that would be generated if the call is exercised at this point ($t=\tau$), the stock acquired at the strike price K and sold forthwith in the market at S_τ.

Therefore, it follows from $C > S_\tau - K$, that the unexercised option would yield a higher profit than the strategy of exercising the option, acquiring the stock and selling the stock in the market. Since τ is an arbitrary time point in (0,T), this argument holds for all $0<\tau<T$.

So it would be better that you sell the American call without exercise at a price C rather than exercise the call, pay a price K, receive the stock and sell it in the market at S_τ.

An important corollary of this result is that because it is never optimal to exercise the American call before maturity, the right embedded in the option due to the flexibility of early exercise carries no worth in the market, carries no price in the market and therefore the American call on a non-dividend paying stock would be priced the same as a European call with the same parameters.