Financial Derivatives and Risk Management
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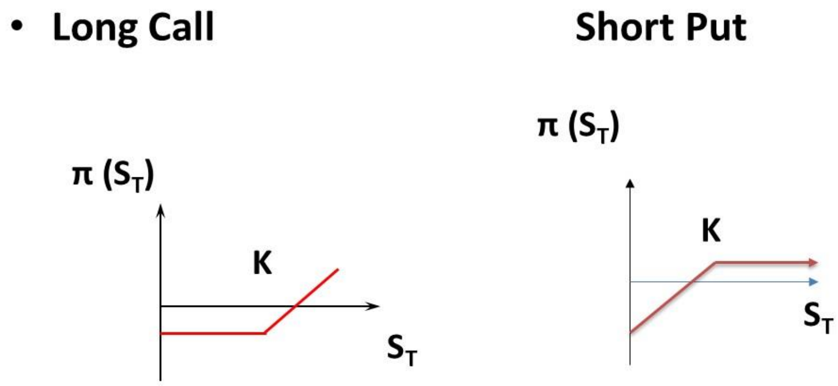
Lecture 31
Options: Put-Call Parity

Long call vs short put



It is clear from the above profit profiles that if an investor takes up a long call or a short put, he makes a profit if the stock price goes up in either case. The profit is unbounded in the case of the long call while being restricted to the premium amount in the short put position. Further, once the stock price at maturity S_T exceeds the threshold of exercise price K, every change in stock price is mimicked by an equal change in the long call payoff. Nevertheless, the payoff from the short put remains unchanged. Let us examine the nature of the payoffs from the two strategies:

(i) A long call entitles the investor to buy the stock at a pre-determined price K. It follows that if the stock price at maturity $S_T>K$, then the investor can invoke the option, buy the stock at K and sell it forthwith in the market at S_T thereby earning a payoff of S_T-K. K is the threshold price. Below K the long party will not exercise the call, it will find more profitable to buy the asset from the market rather than exercising the call and therefore the option will expire worthless.

(ii) Consider, now, the short put. Short put means the investor has written the put. Obviously, because writing the put entails an obligation (the obligation to deliver the stock if the party long in the put decides to exercise it), the writer will receive a premium for undertaking the commitment. If $S_T<K$, then the put holder will exercise the put and the writer will have to deliver the stock with a market worth of S_T at K ($S_T<K$) and therefore the payoff will be negative. If, however, S_T ends up greater than K, then the put holder will not exercise the put and rather sell the stock in the market, letting the option lapse. In this scenario, the put holder gets the unencumbered right to retain the premium that he received when he wrote the put. Thus, if $S_T>K$, the put writer makes a profit equal to the premium on the put. This is obviously constant, received at the time of trading the put, not at maturity.

Now, if we compare these two strategies, the long call and the short put, we find that both of them have a certain underlying perception of the investor that the stock price is going to increase. However, for a put writer, it is in a sense, a passive situation where the profit is independent of the stock price increase beyond a threshold whereas for the call holder it is an active situation where the stock price increase is mirrored in the profit profile. In the case of a short put, the profit arises from the "non-exercise" of the option by the "other" party to the contract. If the stock price is higher than the exercise price, the put holder would rather sell in the market. Therefore, the put option will expire worthless and the put writer will pocket the premium. In other words, the profit that the put writer generates is merely because of the fact that the put holder is not exercising the option. It is a kind of a default situation rather than by design.

Put-call parity for European options

Put-call parity is a parity relationship, an equivalence relationship, between the prices of the put option and the call option at a given instant of time. ***Both options must be on the same underlying having the same maturity and exercise price.*** To start with, we assume that both options are European although the logic can be extended to American options as well.

We arrive at the parity relationship by invoking the condition of no-arbitrage. The strategy is that we construct two portfolios having the same payoffs (with the same level of risk) at a certain same point in time. One of the portfolio comprises of assets whose prices/values are known or easily ascertainable while the other portfolio comprises of assets to be priced. For the purpose of establishing put-call parity we construct our portfolio as follows:

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
BUY CALL	-c	0	S_T-K
SELL PUT	+p	-(K-S_T)	0
INVEST	-Ke$^{(-rT)}$	K	K
TOTAL	p-c+ Ke$^{(-rT)}$	S_T	S_T

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
TOTAL	p-c+ Ke$^{(-rT)}$	S_T	S_T
PORTFOLIO B			
BUY STOCK	-S_0	S_T	S_T
TOTAL	-S_0	S_T	S_T

Now these two portfolios A & B have the same cash flows as on the date of maturity of the options i.e. t=T. They have no cash flows in between in the interval (0,T). We assume that the two portfolios have the same risk profile. Indeed, we assume away the possibility of default risk on the option contracts and the investments is also made in riskfree securities at the riskfree rate. So that makes Portfolio A riskless. As far as Portfolio B is concerned, the cashflows arise from the sale of the stock that is already in the possession of the investor. Hence, these cashflows are also riskfree. Thus, both portfolios A & B carry the same level of risk. It follows from the no arbitrage requirements that they must be priced equally at equilibrium. This gives us the put-call parity relationship:

$c+Ke^{-rT}=S_0+p$

This is a very important relationship in option theory and it is called put-call parity in relation to European options. Remember the relation assumes:

(i) Both options are European style.
(ii) Both options have the same underlying S.
(iii) Both options have the same exercise price K.
(iv) Both options have the same maturity T.
(v) Option contracts are default free.

Put-call parity can be very compactly derived as follows:
Payoff of (Long call+Short Put)= $\max(S_T-K,0)-\max(K-S_T,0)$
$= \max(S_T-K,0)+\min(S_T-K,0)=S_T-K$.
Hence, c-p=Cost of (Long Call+Short Put)=PV of expected payoff=S_0-Ke^{-rT}.

Arbitrage procedure if put-call parity does not hold

How can arbitrage profits be earned if put-call parity is violated? We address this issue. It may so happen that at a particular point in time, due to certain spontaneous factors or certain market aberrations/ anomalies or information distortion or time lag in information dissemination, this relationship may not hold. Let us assume that

$c+Ke^{-Rt}>S_0+p$

subsists in the market at some point in time. We can, then, extract arbitrage profits by the following exercise:

	t=0	t=T	
		$S_T<K$	$S_T>K$
WRITE CALL	+c	0	$-(S_T-K)$
BORROW	$Ke^{(-rT)}$	-K	-K
BUY STOCK	$-S_0$	S_T	S_T
BUY PUT	-p	$K-S_T$	0
TOTAL	$c+Ke^{(-rT)}-S_0-p>0$	0	0
RISKLESS PROFIT $=(c+Ke^{(-rT)})-(S_0+p)$			

At the very outset we find that (S_0+p) is lesser valued compared to $(c+Ke^{-rT})$. It follows logically that we should long the former and short the latter in order to earn arbitrage profits in accordance with the maxim "buy cheaper, sell dearer". Accordingly, we formulate the following strategy:

(i) Borrow an amount Ke^{-rT} @ riskfree rate r for maturity T;
(ii) Write a European call on the same stock S, with same exercise price K and maturity T;
(iii) Buy European put on the stock S with exercise price K and maturity T;
(iv) Buy one unit of the stock S for S_0.

Let us work out the payoff from this strategy at maturity of the options i.e. t=T. We have:

$\Pi\ (S_T)= -K -\max(S_T-K,0) +\max(K-S_T,0)+S_T=-K +\min(K-S_T,0)+\max(K-S_T,0)+S_T=-K +(K-S_T)+ S_T=0$

Thus, the payoff of this strategy at maturity is 0. But what is the cash-flow at setting it up at t=0? We have:

Cashflow at setting up this strategy at t=0 is: $Ke^{-rT}+c-p-S_0>0$. Thus, this strategy gives us a positive inflow at inception equal to $Ke^{-rT}+c-p-S_0$ without carrying any future obligation. This amount $Ke^{-rT}+c-p-S_0$ therefore represents unencumbered arbitrage profit.

Alternatively,

(i) Borrow an amount S_0+p-c @ riskfree rate r for maturity T;
(ii) Write a European call on the same stock S, with same exercise price K and maturity T;
(iii) Buy European put on the stock S with exercise price K and maturity T;
(iv) Buy one unit of the stock S for S_0.

Let us work out the payoff from this strategy at maturity of the options i.e. t=T. We have:

$\Pi\ (S_T)= -(S_0+p-c)e^{rT}-\max(S_T-K,0) +\max(K-S_T,0)+S_T$
$=-(S_0+p-c)e^{rT}+\min(K-S_T,0)+\max(K-S_T,0)+S_T=-(S_0+p-c)e^{rT}+(K-S_T)+ S_T$
$=-(S_0+p-c)e^{rT}+K=(K+ce^{rT})-(S_0+p)e^{rT}>0.$

Thus, although this strategy does not involve any cash investment on the part of the investor, it yields a positive cashflow at maturity of the options. This, therefore, represents the arbitrage profit. It may be noted that in this case the arbitrage profit arises at t=T i.e. on maturity of the options while in the previous case the arbitrageur got the profit at t=0.

In the other case, $c+Ke^{-Rt}<S_0+p$, the same logic will hold except for the fact that all the above trades will be reversed and the profit will be $c+Ke^{-Rt}-(S_0+p)$.

The case $c+Ke^{-Rt}>S_0+p$ is called cash and carry arbitrage since we buy the underlying asset and hold it with us over the life of the options, while the case $c+Ke^{-Rt}<S_0+p$ is termed as reverse cash & carry arbitrage.

In other words, this difference is simply a riskless profit. If the above strategies are appropriately followed, the arbitrageur ends up with a positive cash flow either at t=0 or at t=T (the options' maturity) without any compensating cash outflow at the other end.

<u>Put-call parity with dollar return</u>

We, now, consider the case when the stock that constitutes the underlying asset for the two options pays off a certain dividend, say D_τ at $t=\tau\in(0,T)$ i.e. during the life of the options. Let D_0 be the present value of the dividend at t=0 at the riskfree rate and D_T be its future value at t=T at the same rate. In this scenario, holding the stock during the life of the option entitles the holder of the stock to this additional cash inflow of D_τ at $t=\tau\in(0,T)$. We formulate the following strategy to establish put-call parity:

(i) Borrow an amount Ke^{-rT} @ riskfree rate r for maturity T;
(ii) Write a European call on the same stock S, with same exercise price K and maturity T;
(iii) Buy European put on the stock S with exercise price K and maturity T;
(iv) Buy one unit of the stock S for S_0.

Let us work out the payoff from this strategy at maturity of the options i.e. t=T. We have:

$\Pi(S_T) = -K - \max(S_T-K,0) + \max(K-S_T,0) + S_T + D_T = -K + \min(K-S_T,0) + \max(K-S_T,0) + S_T + D_T = -K + (K-S_T) + S_T + D_T = D_T$

Thus, the payoff of this strategy at maturity is D_T. Therefore, the cash outflow at t=0 for setting up this strategy must be the present value of D_T at t=0 i.e. D_0.

But what is the cash-flow at setting it up at t=0? We have: Cashflow at setting up this strategy at t=0 is: $Ke^{-rT}+c-p-S_0$. Hence, we must have:

$Ke^{-rT}+c-p-S_0=-D_0$ or $c+Ke^{-rT}=p+S_0-D_0$

This is the modified put-call parity in context of an asset that generates dollar (money) income by carrying it over the life of the options.

Please note that in this case we have modified our strategy/portfolios vis a vis the original put-call parity. In fact, it does not really matter at all. Actually the portfolios A and B are not sacrosanct. In the earlier case, in portfolio A, we had call long call and short put and in portfolio B, long stock and borrowings. We can always migrate algebraically between these two portfolios A and B i.e. we can shift an asset from portfolio A to B or vice versa but with a change in sign so that a long position in an asset in portfolio A is equivalent to a short position in the same asset in B and vice versa. Thus, the long call and short put in A can be moved to B as short call and long put. Now, all the four constituents are resided in the same portfolio. Because we have an equality between the two portfolios, the ultimate result does not change by such transpositions.

The observation from the modified put-call parity $c+Ke^{-rT}=p+S_0-D_0$ is that when the underlying pays a dividend during the life of the options, its effective cost of carrying decreases. This happens because a part of the carrying cost (equal to the present value of dividend D_0) can be financed against the expected dividend inflow. Thus, if S_0 is the stock price at t=0, an amount D_0 therefrom can be borrowed out and out against the projected dividend and repaid when the dividend is actually received at $t= \tau$. Hence, the amount to be financed against the stock is S_0-D_0 and it is the future value of this amount that needs to be repaid at option maturity.

Because the mere holding of the asset yields a cash inflow, the investor can borrow the present value of that inflow straight away and use it to part finance S_0. Thus, the effective cost of carrying the asset falls by D_0 to S_0-D_0.

It may be noted that in this arbitrage, there do exist intermediate cash flows (dividend) but they are accounted for through their present value at t=0.

A different set of arbitrage portfolios gives the same result:

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
BUY CALL	$-c$	0	S_T-K
INVEST	$-Ke^{(-rT)}-D_0$	$K+D_T$	$K+D_T$
TOTAL	$-c-Ke^{(-rT)}-D_0$	$K+D_T$	S_T+D_T

	t=0	t=T	
PORTFOLIO A		$S_T<K$	$S_T>K$
TOTAL	$-c-Ke^{(-rT)}-D_0$	$K+D_T$	S_T+D_T
PORTFOLIO B			
BUY STOCK	$-S_0$	S_T+D_T	S_T+D_T
BUY PUT	$-p$	$K-S_T$	0
TOTAL	$-S_0-p$	$K+D_T$	S_T+D_T
$c+Ke^{(-rT)} = S_0-D_0+p$			