Financial Derivatives & Risk Management
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Lecture 27 - Tailing the Hedge; Clean & Dirty Price

Tailing the hedge

Tailing the hedge involves an adjustment to the hedge ratio, as we calculated on the basis of minimization of variance. Tailing adjustment is required because futures contracts give rise to mark to market settlement, thereby requiring financing of cash outflows or reinvestment of cash inflows. As a consequence of "marking to market" of the futures position, the cash flows occur on a daily basis. The long party's margin account gets debited with decreases in the futures' price over a daily time horizon and credited with the increases. Now, if there is a debit or credit in the margin account, it entails a cost. Of course, the cost may be positive or negative. A debit will call for financing of the shortfall while a credit will require reinvestment thereof. Thus, there arises the need to make an adjustment to the hedge ratio to accommodate this financing cost of these cash flows related to the margin account arising from the "marking to market" effect. This is called tailing the hedge.

An alternative explanation of tailing the hedge can be given in terms of the relative volatility of the spot and futures prices. Consider the following expression which is the equilibrium price relationship between the spot and futures prices viz $F = Se^{rT};\ \Delta F = e^{rT}\Delta S$. This implies that a change in the spot prices ΔS gets magnified to $\Delta F = e^{rT}\Delta S$ in the futures market. Note that $e^{rT} > 1$ for all positive r. Thus, futures prices show greater amplitude of fluctuation and hence greater volatility than the spot prices. Marking to market translates these magnified price changes into mismatched cash flows. Thus, a reduction in the size of the futures position relative to cash is required for maintaining hedge efficiency. This is another interpretation on why tailing of the hedge is required.

Let us now look at the theory behind it. Let us assume that the hedger needs to hedge one1 unit of the cash commodity with a hedge ratio h. Since $h=Q_f/Q_s$, it follows that he takes up a position of h units in the futures market for the hedging.

Thus, the hedged portfolio consists on one unit of the hedged commodity and a position in h units in the futures (hedging instrument). The value of the portfolio at hedge inception (t=0) is $V_0=S_0+hF_0$ and its value at hedge lifting (t=H) is $V_H=S_H+hF_H$ whence the chance in portfolio value over the hedge period (assuming the hedge ratio to be constant throughout) is $\Delta V=\Delta S+h\Delta F$. The variance of the portfolio is given by $\sigma_P^2 = \sigma_S^2 + h^2\sigma_F^2 + 2h\rho\sigma_S\sigma_F$. Minimizing the variance with respect to h, we get $h = -\rho\frac{\sigma_S}{\sigma_F} = -\beta_{SF}$, ρ, the correlation coefficient is a measure of basis risk.

Now if it is assumed that the hedged commodity and the underlying of the futures is the same and ρ=1, then the portfolio variance can be made to vanish by setting h=- ΔS/ΔF and a perfect hedge is achieved.

Now we make the important assumption that the interest rates are non-stochastic. In other words, they are evolving in a predictable manner or that they can be estimated at t=0 with precision. There is no unpredictability about interest rates.

Let d_t be the daily interest rate, compounded daily, applicable for day t to day (t+1). Then the future value of one money unit invested at time t at the maturity date of futures (T) and the end of the hedge period (H) will be:

$$RF_t = (1+d_t)(1+d_{t+1})(1+d_{t+2})...(1+d_{T-1})$$
$$RH_t = (1+d_t)(1+d_{t+1})(1+d_{t+2})...(1+d_{H-1})$$
$$\frac{RF_t}{RH_t} = (1+d_H)(1+d_{H+1})...(1+d_{T-1}) = RF_H$$

The third expression holds for all t. The RHS is independent of t so that for any value of t we have:
$\frac{RF_t}{RH_t} = (1+d_H)(1+d_{H+1})...(1+d_{T-1}) = RF_H$.

Let us assume that we use a variable hedge ratio to be set at the time t that operates from t=t to t=t+1 defined by:: h_t=-1/(RF_{t+1}) i.e. $h_t = -1/\left[(1+d_{t+1})(1+d_{t+2})...(1+d_{T-1})\right]$. In other words, at the end of each period t we set a hedge ratio to applicable to the next period (t+1) as above. We keep on adjusting our position in the futures market such that we achieve this hedge ratio. Instead of a passive strategy of having a hedge ratio set at t=0 that holds for the entire hedge period, we have a variable hedge ratio. In the case of daily MTM settlements, we change the hedge ratio at the end of each day as per the expression: $h_t = -1/\left[(1+d_{t+1})(1+d_{t+2})...(1+d_{T-1})\right]$. This hedge ratio holds upto the next day's settlement. Next day again after settlement, the hedge ratio is modified as per this formula to hold for the following day and so on.

Recall that interest rates are non-stochastic in this model, so the values that enter the hedge ratio at t are known with certainty at that time i.e. RF_{t+1} is known at t. Therefore, this is a practically implementable strategy.

Now, for t=0, we have $h_0 = -1/RF_1 = -1/\left[(1+d_1)(1+d_2)...(1+d_{T-1})\right]$. The MTM transfer at the end of day 1 will be $h_0(F_1-F_0)$ since this hedge ratio will hold until settlement on day 1. Since this would be invested at t=1, its future value at the end of hedge period $h_0(F_1 - F_0)RH_1$**.** Using h_0=-1/(RF_1): we get the future value of the reinvested margin as $-(F_1 - F_0)\frac{RH_1}{RF_1} = -(F_1 - F_0)\frac{1}{RF_H}$.
Summing this over the entire life of the hedge, we get the aggregate future value of all margin transfers on the hedge upto the lifting of the hedge as:
$\pi = -(F_1 - F_0)\frac{1}{RF_H} - ... - (F_H - F_{H-1})\frac{1}{RF_H} = (F_0 - F_H)\frac{1}{RF_H}$. But the hedger will also get a cash inflow of S_H on selling the hedged item in the spot market when he lifts the hedge. Therefore, the

total payoff from the disposal of the entire hedged portfolio (hedged item+hedging instrument) at lifting of the hedge is:

$$\Pi = -\left(F_1 - F_0\right)\frac{1}{RF_H} - \ldots - \left(F_H - F_{H-1}\right)\frac{1}{RF_H} + S_H = \left(F_0 - F_H\right)\frac{1}{RF_H} + S_H .$$

From the cost of carry relationship:

$$F_t = S_t\left(1+d_t\right)\left(1+d_{t+1}\right)\left(1+d_{t+2}\right)\ldots\left(1+d_{T-1}\right) = S_t RF_t;\ F_H = S_H RF_H;\ F_0 = S_0 RF_0 \text{ so that}$$

$$\Pi = \left(F_0 - F_H\right)\frac{1}{RF_H} + S_H = \left(S_0 RF_0 - S_H RF_H\right)\frac{1}{RF_H} + S_H = \frac{S_0 RF_0}{RF_H} = S_0 RH_0$$

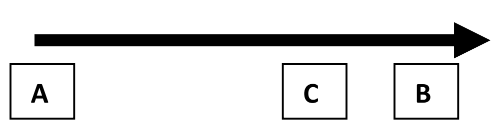
Recall that the cash and carry relationship mandates that the futures price of an asset "at maturity of the futures" be equal to the future value of its spot price at that time. It is emphasized that the cash & carry relationship requires that $F_t = S_t\left(1+d_t\right)\left(1+d_{t+1}\right)\left(1+d_{t+2}\right)\ldots\left(1+d_{T-1}\right)$ hold over the entire life of the futures and not over the hedge period, since this cash & carry principle operates due to the convergence of spot and futures price at maturity of futures. Thus, the expression $F_t = S_t\left(1+d_t\right)\left(1+d_{t+1}\right)\left(1+d_{t+2}\right)\ldots\left(1+d_{T-1}\right)$ is over the entire life of the futures. It is not confined to the hedge period. It has nothing to do with the hedge. No arbitrage operates due to convergence of spot & futures at futures maturity.

Thus, the payoff of the hedged portfolio at lifting of the hedge is S_0RH_0. This is interesting. S_0 is known at inception of hedge period (t=0). Recall that RH_0 is the future value of one unit of money invested at inception of the hedge (t=0) at the end of the hedge period. Since interest rates are predictable, so is RH_0. In other words, the payoff of the hedged portfolio, if the dynamic hedge ratio strategy is implemented as above, is known upfront at hedge inception (t=0) with certainty. The return provided over the hedge period is guaranteed and is therefore the risk free rate. So, by using a dynamic hedge ratio which is set at the beginning of each period, we are able to eliminate the effect of "marking to market".

Thus, if we use a dynamic hedge ratio settled at the beginning of each period given by h_t=-1/(RF_{t+1}), we are assured of a risk free return over the hedged period, notwithstanding the MTM cash flows in the futures contract.

Clean & dirty price of a bond

In the context of bonds, we have 2 prices (i) the clean price; and (ii) the dirty price. The clean price is also called the quoted price and the dirty price, the full price or transaction price. To understand the difference between the two prices, let us look at the time line AB.



Consider a bond that has made a coupon payment at A while the next coupon payment is at B. Let the bond be traded between two parties X(seller) & Y(buyer) at an arbitrary point C in AB. At the outset, it is not at all mandatory that the bond be traded only at the coupon dates e.g. A,B; it can be traded at any time during its lifespan. Now, when the bond is traded at the point C i.e. between two coupon dates, then it would have been held for part of the period AC between these two dates AB by the seller X and for the remaining part BC of the period by the buyer Y. It follows that the interest entitlement for the period AB vests with X and for the period BC with Y. However, when the bond issuer releases the coupon payment at B, it will not split up the interest into that relating to AB and BC separately and pay out to the respective parties. The bond issuer will pay the interest for the entire period AB to the party whose name appears in its register of bondholders on the record date, which would if the transaction takes place before the ex-interest date be the buyer Y. It is, therefore, the norm that when Y acquires the bond from X at C, he compensates X for the interest for the period AC as an add-on over the quoted price of the bond. This add-on is called the accrued interest. The total price i.e. the quoted price plus the add-on of accrued interest is called the dirty price, cash price or transaction price. The quoted price is also called the clean price.

This is the difference between clean and dirty price. The clean price is the bond price without the accrued interest and when we add the accrued interest, we get the transaction price, that is the price at which the actual trade occurs.

In a nutshell, when a bond is sold between two coupon dates, one of the cash flows, the very next cash flow, encompasses two components as shown below:

(i) Interest earned by the seller for the period for which the bond was held by him;
(ii) Interest earned by the buyer for the period the bond was held by him.

The interest earned by the seller is called accrued interest and equals the interest that has accrued between the last coupon payment date and the settlement date. At the time of purchase, the buyer must compensate the seller for the accrued interest. The buyer recovers the accrued interest when the next coupon payment is received.

In general, Cash (Transaction) price = Quoted price + Accrued interest since last coupon date

Example

Suppose that it is March 5, 2018, and the bond under consideration is an 11% coupon bond maturing on July 10, 2025, with a quoted price of $95.50. Coupons are paid semiannually and the final coupon is at maturity, the most recent coupon date is January 10, 2018, and the next coupon date is July 10, 2018. Calculate the dirty price and accrued interest.

Solution

Days: Jan 10, 2018 -Mar 5, 2018 (Actual Period) = 54
Days: Jan 10, 2018 –July 10, 2018 (Ref Period) = 181
Coupon payment for reference period: 5.50

Using Actual/Actual convention: Accrued Interest= 54 X 5.50/181 = 1.64. The cash price per $100 face value for the bond is therefore $95.50 + $1.64 : $97.14

Cash price

When the price of a bond is computed using the present value calculations, it is computed with accrued interest embedded in the price. This price is referred to as the full price. It is the full price that the buyer pays the seller. From the full price, the accrued interest must be deducted to determine the clean (quoted) price of the bond.

FACE VALUE	**1000**
COUPON	**10%**
RATE OF RETURN	**12%**
NEXT & LAST COUPON	**31 12 2019**
01 01 2019	**982.1429**
01 04 2019	**1010.367**
01.10.2019	**1069.272**

Let us look at an example. Let us assume that there is a 10% annual coupon bond of face value 1000. Let us further assume that the market interest rate corresponding to that risk class is 12% p.a. Let there be only the last coupon and principal be outstanding on the bond to be paid on 31.12.2019. Thus, the bond will make a payment of 1,100 on this date.

Let us assume that today is 01.01.2018 i.e. immediately after the previous coupon was paid (31.12.2018). When I price the bond, I will discount the 1,100 to be received on 31.12.2019 for one year with a discount rate of 12%. I arrive at a valuation of 982.

Now, suppose I do the valuation at 01.04.2019. In this case, I will discount 1,100 for 9 months @ 12% (assuming that market rates have not changed. The price works out to 1,010. Why is this difference? Firstly, the amount of cash flow being discounted is clearly unchanged. However, the proximity of the valuation point to that cash flow has decreased by 3 months.

In other words, while the interest for the entire period since the last interest payment date (1.1.2019-31.12.2018) is being discounted, but, in the second case the period of discounting (1.4.2019-31.12.2019) is only part of that period. Thus, the discounted cash flow method of valuation does not exclude the interest for the period which has passed since the last coupon payment when doing the valuation at a point between two coupon dates. The cashflow includes the coupon for the entire period since the last coupon date irrespective of the point between the two coupon dates at which the valuation is done.

The coupon discounted is for the entire period between the two consecutive coupons, but the period of discounting is only for the period remaining till the date of next coupon payment from the date of transaction. In other words, the interest for the period that has elapsed since the last coupon payment is not excluded, not deducted. The correction for this is not made when we work out the cash flow based pricing.

The cash flow that is discounted includes the full coupon amount irrespective of the date of discounting. In other words, while the date of coupon payment becomes closer, the amount discounted remains the same. It is not reduced for the period that has passed. For example, when we price the bond on 01/04/2019, the entire coupon for the period 1.1.2019 to 31.12.2019 is discounted although 3 months have already passed since the last coupon. Coupon is not proportionately reduced.

So in that sense the cash flow pricing gives you the dirty price. It does not give you the clean bond price. It gives you the price with embedded interest.

Formula for the cash price

Now how does the market price a bond? The following is the formula:

$$P_{transaction} = \sum_{t=1}^{n} \frac{C}{(1+r)^{v}(1+r)^{t\text{-}1}} + \frac{M}{(1+r)^{v}(1+r)^{n\text{-}1}} = \frac{1}{(1+r)^{v}}\left[C + \frac{C}{1+r} + ... + \frac{C}{(1+r)^{n\text{-}1}} + \frac{M}{(1+r)^{n\text{-}1}}\right]$$

It is obviously a cash flow formula. If you look at the formula carefully, the bond is priced first at the next coupon date including the next coupon. This is the term in the square brackets. So, suppose the next coupon is 4 months away, then the market first prices the bond at t=4 months by discounting all the future coupon payments (including the coupon payable at t=4 months) and the principal repayments.

In other words, what you get in the square brackets is the price of the bond at t=4 months (cum interest receivable at t=4 months). Next, the market discounts this value at t=4 months to t=0 by using the 4 month discounting rate and arrives at the today's (t=0) price.

If you look at this equation you find that while working out the price the entire next coupon i.e. the coupon receivable at t=4 months (but relating to the full 6-month period i.e. from t=-2 months to t=4months) is being discounted. However, the discount factor (outside the square brackets) is for that period which remains to the next coupon (i.e. t=0 to t=4 months). In other words, what I am trying to say is that the interest that is discounted i.e. that is embedded in this formula relates to the full period for which the coupon applies notwithstanding the trade date being anywhere within the two coupon dates.

Therefore, the price that we get includes the accrued interest. Hence, it is the full price or the cash price and not the clean price of the bond that we arrive at by this cash flow discounting formula.

The price calculated from the present value formula gives you the dirty price value on the valuation rate. Clean price is the quoted price of the bond. Dirty price is the actual price paid by the purchaser. Dirty price is equal to clean price plus accrued interest.