Financial Derivatives and Risk Management

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Lecture 22: Duration and Price Sensitivities, Immunization

I, now, take up the other measures of interest rate risk:

Modified duration

$$Mod\ \ Duration\left(D_{Mod}\right)=-\frac{P'\left(y_0\right)}{P\left(y_0\right)}=\frac{\frac{1}{\left(1+y_0\right)}\sum_{t=1}^{T}\frac{tC_t}{\left(1+y_0\right)^t}}{P\left(y_0\right)}=\frac{D_{Mac}}{\left(1+y_0\right)}$$

In terms of Modified Duration, we have $\frac{dP}{P}=-D_{Mod}dy$. Modified duration is very similar to duration except for the fact that it absorbs the factor $(1+y_0)$ within the definition of duration itself. In other words, it is given by the Macaulay's duration D_{Mac} divided by $(1+y_0)$. The percentage price change, then, becomes: $\frac{dP}{P}=-D_{Mod}dy$.

Interest Rate Elasticity

Interest Rate Elasticity is given by the ratio of the percentage change in price and the percentage change in YTM i.e.

$$IE\left(y_0\right)=\frac{dP/P_0}{dy/y_0}=\frac{P'\left(y_0\right)}{P\left(y_0\right)}y_0=-D_{Mac}\left(\frac{y_0}{1+y_0}\right)=-D_{Mod}y_0\ \sin ce\ \ P'\left(y_0\right)=-D\frac{P\left(y_0\right)}{\left(1+y_0\right)}$$

This is the concept of interest rate elasticity. I repeat, it is the percentage change in price divided by the percentage change in YTM that relates to the given percentage change in price that is being looked at.

It needs be emphasized that in the above analysis we have expressed the percentage change in price in terms of the duration i.e. we have approximated the region of the yield-price curve lying in (y_0-dy, y_0+dy) by a straight line. This approximation holds only when dy is very small so that the curvature effects of the yield-price curve in this region can be neglected.

Summary of measures of interest rate risk

$$DV01 = -\frac{dP}{dy} = D_{Mod} \times P_0 = \frac{D_{Mac}}{1+y_0} \times P_0$$

$$\left.\frac{\Delta P}{P}\right|_{y_0, P_0} = -D_{Mac}\frac{dy}{(1+y_0)} + C\left(\frac{dy}{1+y_0}\right)^2 = -D_{Mod}\,dy + C\left(\frac{dy}{1+y_0}\right)^2$$

$$D_{Mod} = -\frac{P'(y_0)}{P(y_0)} = \frac{\dfrac{1}{(1+y_0)}\displaystyle\sum_{t=1}^{T}\frac{tC_t}{(1+y_0)^t}}{P(y_0)} = \frac{D_{Mac}}{(1+y_0)}$$

$$IE(y_0) = \frac{dP/P_0}{dy/y_0} = -D_{Mac}\left(\frac{y_0}{1+y_0}\right) = -D_{Mod}\,y_0$$

Immunization

Consider an investment in a bond with an investment horizon that does not match the bond's maturity. Suppose at some instant during your holding period, there occurs an unanticipated upward shift in interest rates. Two effects will come into play:

(i) The investor will henceforth be able to reinvest the coupon payments that he receives at a higher rate because the market interest rates have risen. This will enhance the investor's reinvestment income at the end of the holding period. This effect, thus, operates to the benefit of the investor.

(ii) However, when the interest rates increase, the market prices of the bonds will fall. Thus, when the investor disposes of the bond in the market at the end of his holding period, he will get a lower price. This is because prices and interest rates are inversely related and if there is an unanticipated increase in interest rates, it will reflect as a fall in the bond's prices. Thus, this effect will operate to the detriment of the investor. The capital gain would be reduced.

A similar scenario operates when there occurs an unanticipated fall in interest rates. Thus, there are two opposing effects that come into being on the unanticipated change in underlying interest rates. One is the reinvestment income that moves in tandem with the change in interest rates and the other is the capital gains that move opposite to the interest rate change.

Now, both these effects depend on the holding period after the interest rate change. Because larger this remaining holding period, greater is the effect on reinvestment income and lesser on the price at disposal i.e. capital gains and vice versa. This is obvious, larger the residual holding period, larger the number and the period for which coupons will be reinvested and smaller will be the number of coupons and the period for which discounting will be done to arrive at the price at the point of disposal. In other words, what happens is that these two effects not only operate in opposite directions but they are also related to holding period after the interest rate change.

In fact, there exists a unique holding period between t=0 and the maturity of the bond t=T, such that if an investor holds the instrument for that period, these two effects counter balance each other. If these two effects counter balance each other, then, if there is a change in interest rate, the investment gets immunized i.e. the bond portfolio gets insulated from the effects of this interest rate change. This is the philosophy behind immunization.

Immunization: Mathematical formulation

We start with the fundamental pricing equation for a bond:

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

Let us assume that the investor has a holding period (investment horizon) from t=0 to t=H and let the coupons received during this period be reinvested at the current market rate, i.e. y. Then, the total cash flows that the investor will obtain at the end of his holding period will be:

$$TCF(y,H) = \text{Proceeds of Reinvested Coupons} + \text{Sale Price at } t = H : \sum_{t=1}^{H} C_t (1+y)^{H-t} + P_H$$

$$But\ P_H = \sum_{t=1}^{T-H} \frac{C_{H+t}}{(1+y)^t} = \sum_{w=H+1}^{T} \frac{C_w}{(1+y)^{w-H}} = \sum_{w=H+1}^{T} C_w (1+y)^{H-w}$$

$$so\ that\ TCF(y,H) = \sum_{t=1}^{H} C_t (1+y)^{H-t} + \sum_{w=H+1}^{T} C_w (1+y)^{H-w} = \sum_{t=1}^{T} C_t (1+y)^{H-t}$$

$$Hence,\ \frac{\partial TCF}{\partial y} = \sum_{t=1}^{T}(H - t)C_t(1+y)^{H-t-1}\ whence$$

$$\frac{\partial TCF}{\partial y} = 0 \Rightarrow H(y) = \frac{\sum_{t=1}^{T} tC_i(1+y)^{-t}}{\sum_{t=1}^{T} C_t(1+y)^{-t}} = \frac{\sum_{t=1}^{T} tC_i(1+y)^{-t}}{P_0} = D$$

In the above, P_H is the projected price of the bond at the end of the holding period. As usual, it will be equal to the present value of all future cash flows that will be received from the bond henceforth i.e. after t=H.

The result is that if the holding period of the investor coincides with the duration of the bond, then the total cash flows to the investor at the end of his holding period become independent of the market interest rates. This is the mathematical prescription for immunization.

The caveat against immunization

Now, let us look at the assumptions or caveats while using the above formulated immunization framework. The first thing to remember is that we have used Newtonian differentiation. Newtonian differentiation assumes that around that point of reference e.g. y_0, our yield price curve is straight line. This approximation holds only if the region of the yield-price curve being considered i.e. (y_0-dy,y_0+dy) is very small. Remember the yield-price plot has curvature, and it is only when we consider a small section of the plot that we can approximate it by a straight line. In other words, the immunization of the cash flows extends over an infinitesimally small range y-dy to y+dy of interest. This is because Newtonian differentiation assumes a straight line structure around the neighbourhood of the point at which the curve is differentiated. However, because, the price/yield curve is essentially curved, the immunization will not extend over a large variation in y. In other words, if you, say, change an interest rate from say 5% to 9%, this immunity is not going to be there. The portfolio value is going to change even if you hold it for the duration. But, for a small change in the interest rates, very small changes in interest rates; this immunity will hold as I will illustrate by an example to follow.

Furthermore, it is also assumed that the value of y is the same for all maturities i.e. the term structure is flat. In other words, all cash flows that are arising from the instrument are discounted or compounded at the same rate.

It is easy to see that the second derivative $\frac{\partial^2 TCF}{\partial y^2}$ is positive. We have:

$$We\ have\ TCF(y,H)=\sum_{t=1}^{T} C_t(1+y)^{H-t}\ so\ that$$

$$\frac{\partial TCF}{\partial y}=\sum_{t=1}^{T} C_t(H-t)(1+y)^{H-t-1}\ and$$

$$\frac{\partial^2 TCF}{\partial y^2}=\sum_{t=1}^{T} C_t(H-t)(H-t-1)(1+y)^{H-t-2}$$

$$Now,\ if\ \frac{\partial^2 TCF}{\partial y^2}<0,\ we\ must\ have$$

$$either\ (i)\ H>t\ and\ H<t+1\ or\ (ii)\ H<t\ and\ H>t+1$$

$$Now,\ (i)\ cannot\ hold\ because\ H\ must\ be\ a\ positive\ integer\ and$$

$$there\ is\ no\ integer\ between\ t\ and\ t+1.$$

$$(ii)\ is\ obviously\ impossible,\ since\ all\ quantities\ must\ be\ positive$$

$$However,\ \frac{\partial^2 TCF}{\partial y^2}>0\ can\ hold.$$

This shows that the total cash flow at the point t=H is a minima.

EXAMPLE

I have considered a 12% bond with a maturity of 25 years. At a YTM of 16.65%, the Macaulay duration works out to exactly 7 years. Hence, I have fixed the holding period at 7 years and worked out the Total Cash Flows at 7 years for different reinvestment rates: The results are as follows:

R (%)	TCF	R(%)	TCF
11	2251	16	2137
12	2210	16.65	2135.63
13	2180	17	2135.94

14	**2158**	**18**	**2141**
15	**2144**	**19**	**2150**

$$D(16.65\%) = \frac{\sum_{t=1}^{25} tC_t (1+0.1665)^{-t}}{\sum_{t=1}^{25} C_t (1+0.1665)^{-t}} = \frac{\frac{1\times120}{1.1665} + \frac{2\times120}{1.1665} + \ldots + \frac{25\times1120}{1.1665^{25}}}{\frac{120}{1.1665} + \frac{120}{1.1665} + \ldots + \frac{1120}{1.1665^{25}}} = 7 \;\; years$$

Therefore, I fix the holding period at 7 years and work out the total cashflows at the end of this holding period of 7 years corresponding to different reinvestment rates. For example, if the market (reinvestment) rate is 11%, I get

$$TCF(11\% y, 7 \;\; years) = \sum_{t=1}^{25} C_t (1+0.11)^{7-t}$$

$$= 120(1.11)^6 + 120(1.11)^5 + \ldots + 120(1.11)^0 + \frac{120}{1.11} + \frac{120}{1.11} + \ldots + \frac{1120}{1.11^{18}} = 2251$$

Similarly, the other cashflows are computed. We find that if the market rate turns out to be 11% total cash flows turn out to be 2251, if it is 12% then 2210 and so on. At a market rate of 16%, the cashflow is 2137 and at 16.65% it is 2135.63. Again, when this rate increases to 17%, the cashflows become 2135.94. 41. Thus, in the close proximity of 16.65%, the plot of total cashflows against market rates yields a flat region.

The second thing is, at the YTM corresponding to the duration 7 years i.e. at the YTM of 16.65%, the total cashflow is a minimum. It increases on either side of this YTM of 16.65%. So, this is the relevance of duration. It reduces the interest rate risk, the closer is the match between your holding period and the duration of the portfolio, the lesser is the exposure to interest rate risk subject to small changes in interest rates. To summarize, then,

:

(i) The YTM yields the minimum total cash flows for the holding period equal to the duration. Please note that this is the way it should be because duration constitutes the minimum risk holding period also.

(ii) That for interest rates close to the YTM e.g. 16% and 17% the variation in cash flows is very small if the bond is held for its duration.

Let us try to understand what we have done.

1. We started with a particular interest rate i.e. y=16.65%. This is supposed to represent the t=0 interest rate.

2. Step 1: We, then worked out the total cash flows (TCF) for an arbitrary holding period, say D i.e. TCF(y,D)

3. Step 2: We equate the first derivative of this total cash flow TCF(y,D) with respect to y to zero and get an expression for D. For the given bond we find that D=7 years.

4. What does all this mean? It means that if we use y=16.65% and D=7 years, the derivative of TCF with respect to y will be zero. This means that:

(a) the TCF value that we get i.e. TCF(16.65%,7 years) is independent of "SMALL" changes in interest rates around 16.65%, since the derivative is with respect to interest rates.

Let us say that you have taken the position in the bond at t=0. Obviously, P_0 is fixed being the price at which you have bought the bond. If the TCF at t=D=7 years is immunized (as above), then any small change in interest rates will not change the TCF at t=7 years i.e. duration. This means that cash flows at t=0 and t=7 are both fixed and independent of small changes in y. Thus, the return is independent of small changes in y if you hold the bond for 7 years i.e. D. Hence, it will remain at the original y=16.65%.

Suppose the interest rate changes slightly after 1 day after you buy the bond, say at t=1 to say y'=17%. As mentioned above, your return will remain unchanged if you continue holding the bond until 7 years. However, the price of the bond at t= 1 day will readjust so that over the remaining period i.e. t= 1 day onwards, the bond provides a YTM of 17%.

In other words, if another person X buys the bond after the change in interest rate to 17% i.e. after t=1, he will get a return of 17% **if he holds the bond for its duration worked out at 17%** and not 16.65%.

Thus, if you who bought the bond at t=0 when interest rate was 16.65% and sold it to X after the interest rates changed to 17% i.e. after t=1 (but did not hold it until 7 years i.e, D) you will not get the yield of 16.65% but a lower yield.

(b) if there is a large change in interest rates, then this independence will not hold since the flatness of the curve extends only for a small region around 16.65%.

(c) because duration D is y dependent, if we use a different interest rate we shall get a different duration.

(d) If we use a different interest rate for calculating the TCF but calculate the TCF at 7 years, we shall NOT be calculating the TCF at its duration and hence, the immunity will not hold.

(e) because the second derivative of TCF(16.65%,7) with respect to y is positive, it follows that the TCF is a minimum at this t=7 years for y=16.65%. In other words, the TCF will increase on either side of 16.65% but for a holding period of 7 years i.e. if the interest rates change but the holding period is 7 years. But please note that

(i) if the interest rates change significantly, then the duration will no longer be 7 years, it will also change. Thus, the minima at 7 years will also change.

(ii) Also note that TCF may increase but that does not mean that return will increase, return will remain equal to YTM whatever it is. The price at the point at which interest rate make a shift will readjust so that return readjusts to the new YTM.

(iii) Further, the TCF even at 16.65% will gradually increase will passage of time in order to accommodate the return of 16.65%. Important duration relates to interest rate shifts not to time shifts. TCF will naturally increase with time.

One more point, the TCF does increase due to an interest rate shift which may operate to the benefit of the investor. But what if the investor is SHORT in the bond?

In fact, duration is a very useful tool of asset liability management for banks and financial institutions. The closer their assets and liabilities are matched in terms of duration, the lesser is their interest rate risk.

Duration with continuous compounding

$$P(y)=\sum_{t=1}^{T}C_t e^{-yt};\ P'(y)=-\sum_{t=1}^{T}tC_t e^{-yt}$$

$$\frac{dP}{P}=\frac{P'(y)}{P(y)}dy=-\frac{\sum_{t=1}^{T}tC_t e^{-yt}}{P}dy=-Ddy\ \ where\ D=\frac{\sum_{t=1}^{T}tC_t e^{-yt}}{P}$$

This may be compared with the earlier expression $D(y)=\dfrac{\sum_{t=1}^{T}\dfrac{tC_t}{(1+y)^t}}{P}$

Alternative definition of duration

$$\frac{dP}{P}=-D\frac{dy}{1+y}$$

$$d(\ln P)=-Dd\left[\ln(1+y)\right]=Dd\left[\ln\frac{1}{(1+y)}\right]$$

$$D=\frac{d(\ln P)}{d\left[\ln\frac{1}{(1+y)}\right]}=\frac{d(\ln P)}{d(\ln P_1)}$$

$$where\ P_1=\frac{1}{(1+y)}=Current\ price\ of\ a\ one\ year\ zero\ bond\ of\ face\ value\ 1.$$

Duration of zero coupon bonds

The duration of a zero coupon bond equals its maturity. This immediately follows from the fact that such bonds have only one cashflow and that takes place at the maturity of the bond. Hence, longer term zeros are much more price sensitive than short term ones. This makes intuitive sense, since a change in yield for a long term bond affects cash flows over a larger number of discounting periods.

Duration of level coupon bonds

$$P = P_C + P_F$$
$$P_C = \sum_{t=1}^{T} \frac{cF}{(1+y)^t} = cF\left[\frac{(1+y)^T - 1}{y(1+y)^T}\right] = cF\left[\frac{1}{y} - \frac{1}{y(1+y)^T}\right]$$
$$\frac{dP_C}{dy} = -\frac{cF}{y^2} - cF\left[-Ty^{-1}(1+y)^{-T-1} - y^{-2}(1+y)^{-T}\right]$$
$$= -\frac{cF}{y^2} + TcF\,\frac{1}{y(1+y)^{T+1}} + cF\,\frac{1}{y^2(1+y)^T}$$
$$= -\frac{1}{y}cF\left[\frac{1}{y} - \frac{1}{y(1+y)^T}\right] + TcF\,\frac{1}{y(1+y)^{T+1}}$$
$$= -\frac{1}{y}P_C + \frac{Tc}{y(1+y)}P_F$$

$$P_F = \frac{F}{(1+y)^T};\ \frac{dP_F}{dy} = -\frac{TF}{(1+y)^{T+1}} = -\frac{TP_F}{(1+y)}$$
$$D = -\frac{(1+y)}{P}\frac{dP}{dy} = -\frac{(1+y)}{P}\left(\frac{dP_C}{dy} + \frac{dP_F}{dy}\right)$$
$$= -\frac{(1+y)}{P}\left[-\frac{1}{y}P_C + \frac{Tc}{y(1+y)}P_F - \frac{TP_F}{(1+y)}\right]$$
$$= \frac{P_C}{P}\left(1+\frac{1}{y}\right) + \frac{TP_F}{P}\left(1-\frac{c}{y}\right)$$

A fallout of this is that for a perpetuity, the duration would become (1+y)/y since P_F will be zero. Thus, the duration of a perpetuity is independent of the coupon rates and is solely determined by the market rates.

Some properties of duration

Duration & coupon rates

For a given YTM and maturity, duration of all types of bonds (par, premium and discount) would decrease with increase in coupon rate. This is because as coupon increases a greater proportion of the cashflows are realized by the investor earlier.

As an illustration, we consider the case of a 5 year Rs 1,000/- bond with a YTM of 20%. The values of the various measures of interest rate sensitivity are tabulated below:

Coupon Rate (%)	DV01(Rs/%)	D_{Mod} (Years)	D_{Mac} (Years)	Price (Rs)
5	20.03	3.63	4.36	551
10	23.32	3.33	3.99	701
15	26.61	3.13	3.76	850
20	29.90	2.99	3.59	1000
25	33.20	2.89	3.47	1150
30	36.48	2.81	3.37	1299

As is seen in this example, an increase in the coupon rate decreases the Macaulay's duration throughout. As coupon rate increases, the recoupment of the investment is faster and therefore, the duration is less and the sensitivity to the interest rates is also reduced.

EXAMPLE

Two bonds X and Y are 10% and 80% annual coupon bonds of the face value of 1,000. They are redeemed at par after thirty years. Calculate the percentage change in price of each bond when the market interest rates change from 5% to 6%.

$For\ \ Bond\ \ X$

$P = P(10\%, 5\%, 30) = 100PVIFA(5\%, 30)$

$+PVIF(5\%, 30) = 1769$

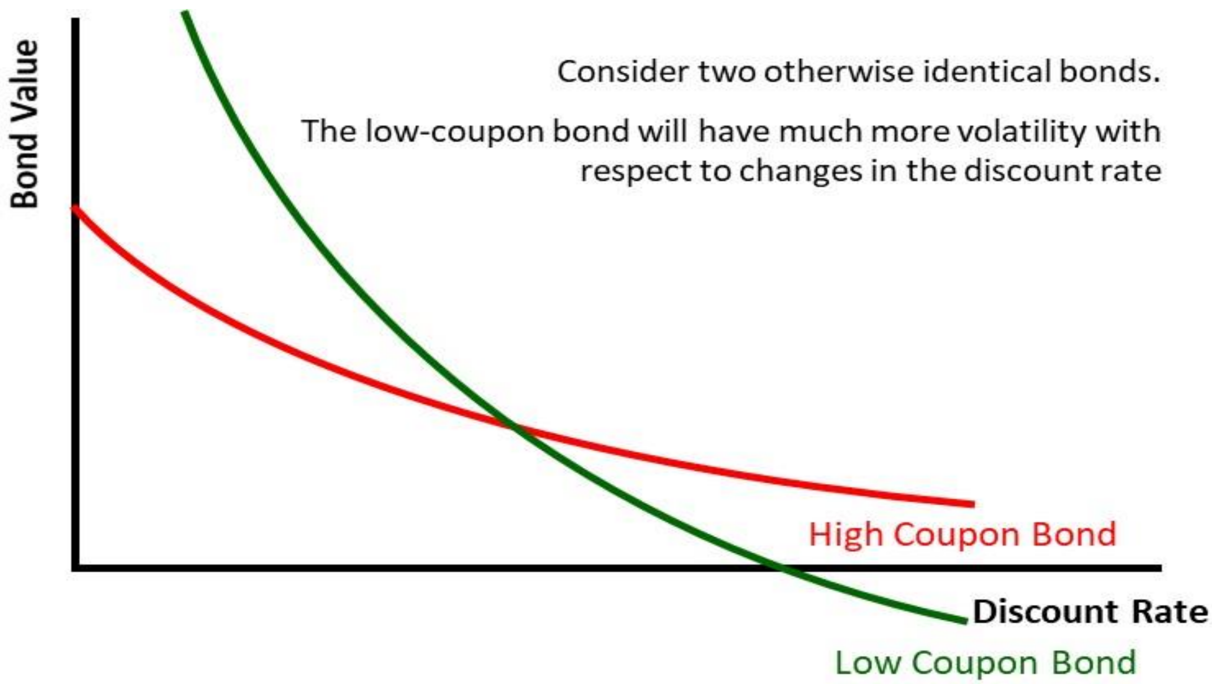
$P^* = P(10\%, 6\%, 30) = 100PVIFA(6\%, 30)$

$+PVIF(6\%, 30) = 1551$

$\Pr ice\ Volatility = \frac{(P^* - P)}{P} = -12.32\%$

$Similarly, \Pr ice\ Volatility\ of\ \ Bond\ Y = -10.72\%$

Thus, as the coupon size of the bond increases, the price sensitivity decreases in magnitude.



Duration & maturity

(i) Duration of par and premium bonds always increases with increasing maturity.

(ii) for discount bonds, there would be a critical value of maturity, upto which duration would increase with maturity. If maturity exceeds this critical value, then duration will start decreasing with maturity for <u>discount bonds</u>.

As an illustration of this phenomenon, we consider a Rs 1,000 face value bond with a coupon rate of 15% quoting at a YTM of 25%. The above data corresponds to a $T_c = 11.5$ years.

Maturity (Years)	DV01(Rs/%)	D_{Mod} (Years)	D_{Mac} (Years)	D*
3	16.63	2.06	2.58	-6.8
5	21.38	2.92	3.66	-5.2
10	24.86	3.87	4.83	-1.2
20	24.46	4.05	5.06	6.8
25	24.21	4.03	5.03	10.8
50	24.00	4.00	5.00	30.8

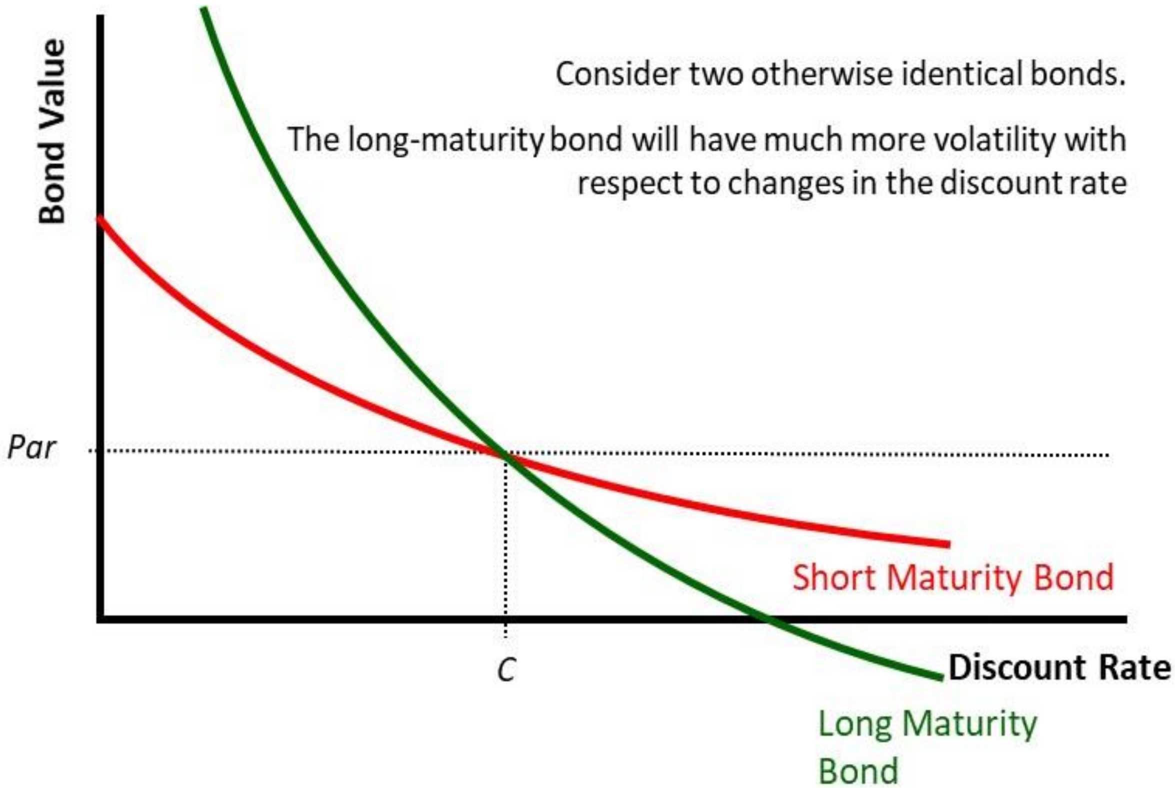
As the maturity increases, duration rises to 5.06 and then it starts declining. It gradually tapers down towards 5.00 which is the duration of a perpetuity.

<u>EXAMPLE</u>

Two bonds X and Y are both 12% annual coupon bonds of the face value of 1,000. They are redeemed at par after two years and ten years respectively. Calculate the percentage change in price of each bond when the market interest rates change from 5% to 6%.

$$\begin{aligned}
&For\ \ Bond\ \ X\\
&P = P(12\%,5\%,2) = 120 PVIFA(5\%,2)\\
&+PVIF(5\%,2) = 1130\\
&P^* = P(12\%,6\%,2) = 120 PVIFA(6\%,2)\\
&+PVIF(6\%,2) = 1110\\
&\text{Pr}ice\ Volatility = \frac{(P^* - P)}{P} = -1.78\%\\
&Similarly,\ \text{Pr}ice\ Volatility\ of\ \ Bond\ \ Y = -6.42\%
\end{aligned}$$

As the maturity of the bond increases, the price sensitivity increases in magnitude.



SUMMARY

(i) Bond prices and market interest rates move in opposite directions.

(ii) When coupon rate = YTM, price = par value.

(iii) When coupon rate > YTM, price > par value

(iv) When coupon rate < YTM, price < par value

(v) For two bonds with same YTM, the lower coupon bond has a higher magnitude of relative price change than a higher coupon bond if the market interest rate changes.

(vi) For two bonds with the same YTM, the bond with longer maturity has higher magnitude of relative (%) price change than one with shorter maturity, if the market interest rate changes.

DV01 & Duration

We have: $DV01 = -\frac{dP}{dy} = D_{Mod} \times P = \frac{D_{Mac}}{1+y} \times P$

Hence, two effects interact due to interest rate shifts viz.

(i) Duration effect
(ii) Price effect

The duration effect of par & premium bonds increases with maturity. Further, the price of a premium bond increases with maturity while that of a par bond remains constant with change in maturity. It follows that DV01 of premium and par bonds will always increase with maturity.
For short term discount bonds, the duration effect dominates and DV01 increases to start with. As maturity increases, the decrease due to price effect also becomes significant and DV01 starts decreasing gradually until a limiting value corresponding to a perpetuity is reached. Thus, in the case of discount bonds, since duration initially increases and then it tends to decrease while price tends to decrease with maturity, we have two conflicting effects for some maturities. For short maturities, the duration effect dominates the price effect and DV01 increases but at longer maturities both the duration and price will decrease with increase in maturities thereby causing DV01 to fall.