**Financial Derivatives & Risk Management**
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**Lec 21 Interest Rate Risk**

Welcome back, in the last lecture I explained the various measures of returns on bond portfolios e.g. yield to maturity, holding period yield and discount yield. While talking about the YTM, I emphasized that there were two inbuilt assumptions in the calculation of YTM viz. (i) that the reinvestment rate equals the YTM and (ii) the bond will be held by the investor till its maturity. Let us, briefly revisit this issue.

Consider an investment in zero coupon bond for a period equal to its maturity. It is quite obvious that the return on this investment is simply determined by its current price i.e. the initial investment. In particular, the redemption value is known at the time of investment by virtue of it being specified in the contract of issue and there are no intermediate cashflows to reinvest. Hence, at any point in time, we can precisely predict/ calculate the return on the basis of the current price. There is no randomness and hence, no risk in the context of such investments. This logic can be extended to coupon paying bonds as well provided we have some mechanism of pre-fixing the rates at which these coupons will be reinvested e.g. by taking forward contracts to fix these reinvestment rates at the point of investment. In such a situation again, the total cashflows at the maturity of the instrument and hence, the return is known with certainty, thereby eliminating any risk.

Indeed, the YTM philosophy premises itself on this logic as it prefixes the reinvestment rate at the YTM and mandates holding the bond to its maturity. It follows that YTM also is uniquely determined by the price at the time of computation and has no element of randomness. It is fixed precisely by the price of the instrument.

However, in the more general scenario, when an investor needs to liquidate the bonds at a point prior to maturity, he needs to do so by selling them in the market at the then prevailing market price. In such situations, his return, obviously, depends on the (i) the reinvestment rates at which coupons are reinvested as well as (ii) the market price at which the liquidation will take place and, thus, becomes susceptible to (i) reinvestment rate risk arising from the uncertainty enshrining the interest rates at which reinvestment could be made and (ii) price risk arising from the possible variability of this market price at liquidation. This variability of the market price arises in consequence of the fluctuations in the underlying interest rates, as there exists a functional relationship between the market interest rates and the prices e.g.

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{\left(1+S_{0t}\right)^t}$$

This impact of return on investment in bond portfolios due to the price fluctuations of the constituent instruments following on changes in market interest rates is called interest rate risk. I take it up in detail in the following:

**Interest rate risk**

The interest rate risk is the risk faced by an investor in bond portfolios arising from the possible changes in the value of his portfolio due to the fluctuations of market interest rates. Since the prices of bonds is a function of market interest rates, the risk that an investor faces when investing in a bond portfolio is that the price of a bond held in the portfolio will decline if market interest rates rise unanticipatedly. This risk is referred to as interest rate risk.

Why do I emphasize “unanticipated”?  Because if the change was anticipated, then it would have been built into the price of the bond itself and as a result, the price would not change because of that anticipated change in interest rates. The impact of the anticipated change is already incorporated in the price of the bond. However, if the change in interest rate is unanticipated by the market, then obviously it will impact the price of the bond.

To reiterate, then, the changes in the value of a portfolio consequent to unanticipated changes in the underlying interest rates, which determine the value of those bonds constitutes the interest rate risk.

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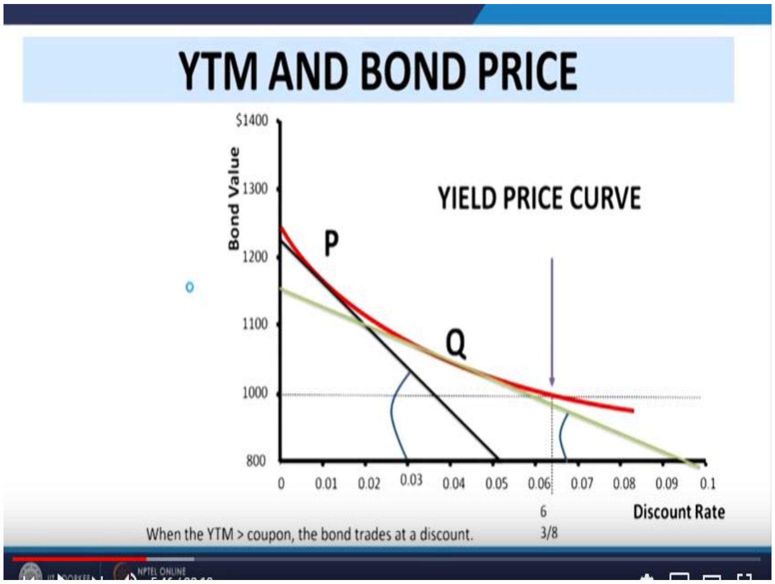


It is clearly seen from the above diagram that there is an inverse relationship between the value of the bond and the interest rates. As the interest rates increase, the value of the bond decreases and vice versa.

**Convexity of yield-price curve**

We have

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \quad so\ that\ \frac{dP_0}{dy} = -\frac{1}{(1+y)} \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t} < 0;$$

$$\frac{d^2 P_0}{dy^2} = +\frac{1}{(1+y)^2} \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^t} > 0$$

(i) From the above, we see that the yield price curve has a negative first derivative establishing the inverse relationship between yield and price. Negative slope also shows that the angle of the tangent with the X-axis lies in the second quadrant i.e. between 90° and 180°.

(ii) The second derivative is positive. This shows that (i) the slope is not constant so that the yield price curve is not a straight line and (ii) the slope increases as the yield increases, but because the slope is throughout negative, an increase of the slope means a decrease in its magnitude as yield increases i.e. the slope moves towards 180° as the yield increases.

From (i) & (ii), it follows that the yield-price curve is convex to the origin as shown below:

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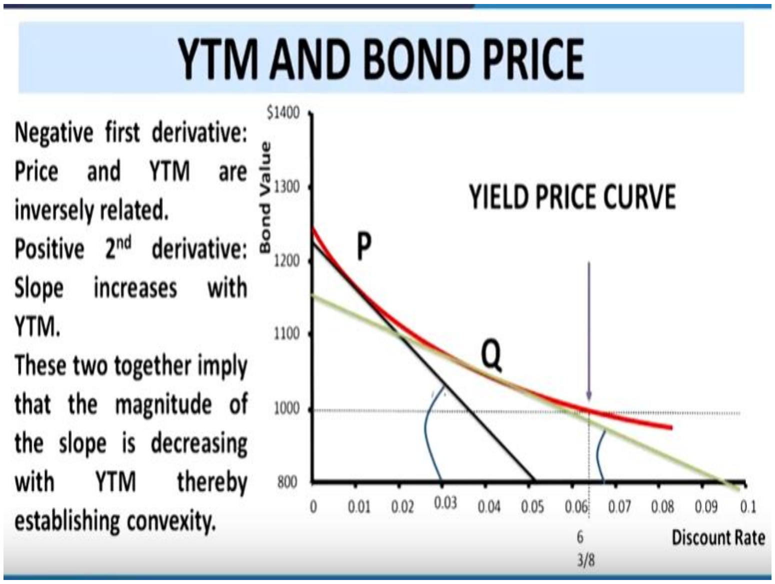


<u>**Impact of convexity of yield-price curve**</u>

Consider a particular yield, say $y=y_0$, at which a bond is priced at $P=P_0$. Let there occur a decrease in yield by, say, an amount $\Delta y$, whence the price increases by $\Delta P_+$. Suppose, now, that the yield had instead of decreasing by $\Delta y$, decreased by $\Delta y$ and the consequential price change was $\Delta P_-$. It, then, follows, as a consequence of the convexity of the yield-price curve that $\Delta P_+ > \Delta P_-$. In other words, if there occurs a certain decrease in yield, then the increase in price is more than the decrease in price corresponding to an increase in yield of the same absolute magnitude.

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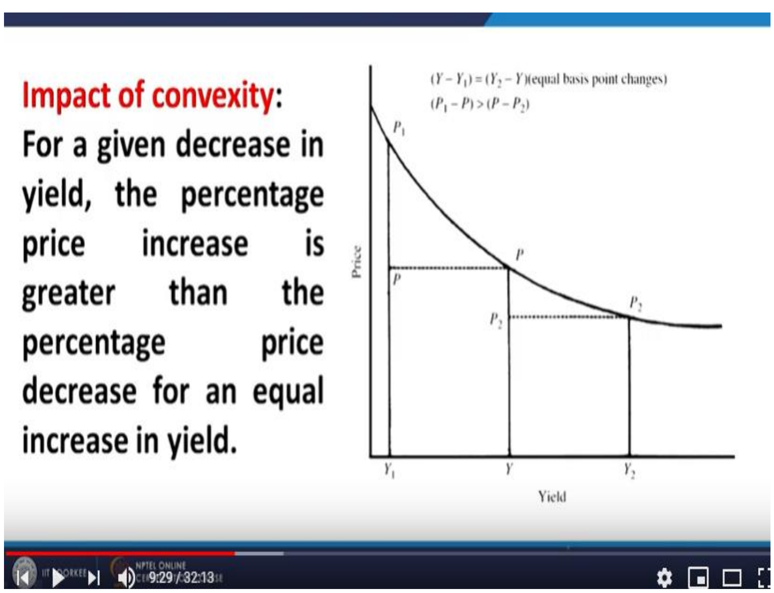


## Measures of interest rate risk

I discuss the following common measures of risk:

(i) Dollar value per basis point (DV01)

(ii) Macaulay's duration

(iii) Modified duration

(iv) Interest rate elasticity

## Dollar value per basis point

Dollar value per basis point (DV01) is the change in bond price corresponding to a change of one basis point in the yield. It is given by the negative slope of the price/yield curve:

$$DV01 = -\frac{dP(y)}{dy}$$

Thus, DV01 is the negative of the slope of the yield price curve. The yield price curve slope is negative as explained earlier so DV01 returns a positive number.

## Macaulay's duration

Macaulay's duration or simply duration is the most important measure of interest rate risk. Let us start with the equation of yield-price curve. Price is equal to the present value of future cash flows discounted at the YTM.

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

Let us say we identify a particular point $y=y_0$, on the price yield curve around which we desire to investigate the impact of the price of small fluctuations in the interest rates. Let the price of the bond at $y=y_0$ be $P_0=P(y_0)$. Let the interest rate change from $y_0$ by an infinitesimal amount dy to $y_0$+dy whence the price changes by dP to $P(y_0+dy) = P_0+dP$. Then we have, by Taylor expansion:

$$P_0 + dP = P(y_0) + dP = P(y_0 + dy) = P(y_0) + P'(y_0)dy + \frac{1}{2}P''(y_0)(dy)^2 + ...$$

$$or\ \left.\frac{dP}{P}\right|_{y_0,P_0} = \frac{P(y_0+dy) - P(y_0)}{P(y_0)} = \frac{P'(y_0)}{P(y_0)}dy + \frac{1}{2}\frac{P''(y_0)}{P(y_0)}(dy)^2 + ....$$

$$= -D\frac{dy}{(1+y_0)} + C\left(\frac{dy}{1+y_0}\right)^2 + ...\ where\ we\ set$$

$$Duration\ D = -\frac{(1+y_0)P'(y_0)}{P(y_0)} = \frac{\sum_{t=1}^{T}\frac{tC_t}{(1+y_0)^t}}{P(y_0)},\ Convexity\ C = \frac{(1+y_0)^2 P''(y_0)}{2P(y_0)} = \frac{\sum_{t=1}^{T}\frac{t(t+1)C_t}{(1+y_0)^t}}{2P(y_0)}$$

It is emphasized that the quantities D (Duration) and C (Convexity) are defined as above by design. The objective of such a definition will become transparent as we proceed. Thus, for the moment, I reiterate that we define Duration as:

$$D = -\frac{(1+y_0)P'(y_0)}{P(y_0)} = \frac{\sum_{t=1}^{T}\frac{tC_t}{(1+y_0)^t}}{P(y_0)}$$

and Convexity as:

$$C = \frac{(1+y_0)^2 P''(y_0)}{2P(y_0)} = \frac{\sum_{t=1}^{T}\frac{t(t+1)C_t}{(1+y_0)^t}}{2P(y_0)}$$

Further, in terms of Duration and Convexity the percentage price change corresponding to an infinitesimal shift in interest rates dy around $y_0$ is given by:

$$\left.\frac{dP}{P}\right|_{y_0,P_0} = -D\frac{dy}{(1+y_0)} + C\left(\frac{dy}{1+y_0}\right)^2$$

It is obvious from the expressions for D & C that they have the dimensions and units of time and $(\text{time})^2$ respectively while dy has the units of % per unit time. $(1+y_0)$ is actually $(1+y_0.1)$ making it dimensionless as $y_0$ is in % per unit time and when multiplied by unit time it yields a dimensionless quantity.

## Duration: a linear approximation of the yield-price curve

Let us, for the moment, ignore the convexity i.e. the P''(y) term in the expression for percentage price change. Then, we have:

$$dP = P'(y_0)dy;\ P'(y_0) = \text{-}D\frac{P(y_0)}{(1+y_0)}\ \text{so that}\ dP = \text{-}D\frac{P(y_0)}{(1+y_0)}dy \propto dy\left(\text{for given } y_0\right)$$

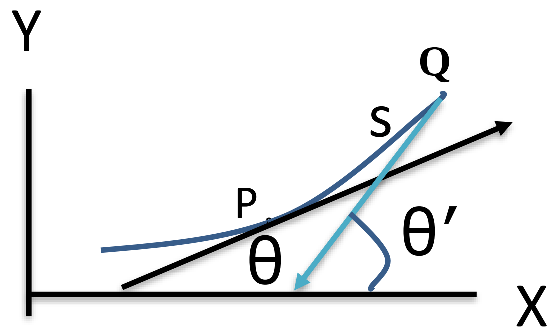
$$\text{Also}\ \left.\frac{dP}{P}\right|_{y_0} = \text{-}D\frac{1}{(1+y_0)}dy \propto dy$$

since $D = \dfrac{\sum_{t=1}^{T}\dfrac{tC_t}{(1+y_0)^t}}{P(y_0)}$ is fixed for a given $y_0$ for a given bond. In other words, if price changes are calculated using only "duration" effect, they will turn out to be proportional to the change in yield that cause those price changes. In fact, whenever we truncate the Taylor series to first order, a straight line approximation of the underlying curve in the infinitesimal neighborhood of the point of reference is implicit. Simply stated, in a very infinitesimal neighborhood of $y_0$, dP (computed through the use of duration) becomes proportional to dy.

Therefore, if we ignore further terms in the Taylor series than the first order, then in the close neighborhood of $y_0$, we implicitly assume that the yield-price curve is a straight line and so the price change in this small (infinitesimal) region around $y_0$ as well as the percentage price change become proportional to dy. However, it does not, in any way, imply that the entire yield-price plot is a straight line. We assume that a small region around $y_0$ to be a straight line.

***The cardinal upshot of the above detailed analysis is that the "duration" approximation would hold only for very small shifts in interest rates. For significant changes in interest rates, the effect of curvature becomes too significant to be ignored.***

I, now, discuss briefly the concept of curvature in two-dimensional space before elaborating the role of convexity in interest rate risk.




Consider a point P on a curve S≡y(x). Let the tangent at P make an angle θ with the X-axis. Let us move a distance δs along the curve S to a point Q, the tangent at which makes an angle θ' with the X-axis. So, as we move along the curve S from P to Q, the tangent which was making an angle θ with the X-axis at P now makes an angle θ' with the X-axis at Q. In other words, the angle that the tangent makes with the X-axis has change from θ at P to θ' at Q. Thus, the slope of the curve S has changed as we move from P to Q along the curve S.

Then, the curvature of the curve S at the point P is defined by the rate at which the tangent line turns as we move along the curve from P to Q in the limit that Q is infinitesimally close to P. In other words, it is the rate of change of this angle θ as we move for an infinitesimal distance along this curve S Mathematically, we define curvature by:

$$\kappa = \lim_{Q\to P}\frac{\theta'-\theta}{PQ} = \lim_{\delta s\to 0}\frac{\delta\theta}{\delta s} = \frac{d\theta}{ds} = \frac{d\theta/dx}{ds/dx} = \frac{d\left[\tan^{-1} dy/dx\right]/dx}{ds/dx}$$

$$\frac{d}{dx}\left[\tan^{-1}\left(dy/dx\right)\right] = \frac{1}{1+\left(\dfrac{dy}{dx}\right)^2}\frac{d^2y}{dx^2};\ \frac{ds}{dx} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$\kappa = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{-3/2}\frac{d^2y}{dx^2}$$

Please note that the rate of change is not with respect to x, but with respect to s, the arc length. i.e. as we move along the curve S.

If we look at the expression for the curvature carefully:

$$\kappa = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{-3/2} \frac{d^2y}{dx^2}$$

The denominator contains dy/dx which is just the slope of the tangent i.e. it is the straight-line approximation at the given point P. When we talk about the slope of the tangent, we simply identify a point Q infinitesimally close to P and then approximate the slope at P by the slope of the straight line PQ. So, the denominator is simply the straight line approximation. Thus, the curvature of the given curve at P is captured by the second derivative at P i.e. $\left.\frac{d^2y}{dx^2}\right|_P$ . The second derivative captures the curvature of the curve S at P.

**Convexity of yield-price curve**

Applying the above explanation of curvature in context of the yield price curve, it follows that the convexity of a bond, defined as it is, in terms of the second derivative P"(y):

$$C = \frac{(1+y_0)^2 P''(y_0)}{2P(y_0)} = \frac{\sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y_0)^t}}{2P(y_0)}$$

captures the curvature effect of the yield-price curve. In essence, convexity may be viewed as a correction to the duration based calculations that adds to the precision of the results by factoring in the effect of curvature into the results. In particular, higher the curvature (convexity) of a bond, greater will be the deviation of its actual price shift due to a shift in interest rates from the shift calculated using the "duration" formula. In other words, greater the convexity, higher will be the error by using the "only duration" formula. If the curve is relatively flat, use of duration alone will give a reasonably good approximation. If the curvature is significant, then the second derivative P"(y) and hence the convexity has a significant value and the deviation of the actual price change from the price change worked out using duration alone (without the convexity correction) would be quite significant. Therefore, convexity becomes very important in such situations.

We conclude that if we use the duration alone for computing the percentage price change, we are approximating the relevant region around our point of reference $y_0$ by a straight line. We are ignoring the effect of curvature. But the yield-price curve is actually curved. It is convexity that captures the curvature of the yield price curve so that if one uses duration plus convexity, we enhance the precision of our computations

**EXAMPLE**

Consider a 12% coupon bond of face value of 100 with a yield to maturity of 18% and 5 years remaining to maturity.

(i) What is the bonds current price, assuming annual coupons?
(ii) What is the bond's Duration? Convexity?
(iii) What percentage price change might you expect if the yield to maturity suddenly increased to 25%? Calculate using Duration alone and then using both Duration & Convexity.
(iv) What would be the exact percentage price change?

The solution of this example is given below:

$$P_{18\%} = \sum_{t=1}^{5} \frac{C_t}{(1+0.18)^t} = \frac{12}{(1+0.18)} + \frac{12}{(1+0.18)^2} + ... + \frac{112}{(1+0.18)^5} = 81.24$$

$$Duration(D) = \frac{1}{P_{18\%}} \sum_{t=1}^{5} \frac{tC_t}{(1+0.18)^t} = \frac{1}{81.24}\left[\frac{1\times 12}{(1+0.18)} + \frac{2\times 12}{(1+0.18)^2} + ... + \frac{5\times 112}{(1+0.18)^5}\right]$$
$$= 3.93\ years$$

$$Convexity(C) = \frac{1}{P_{18\%}} \sum_{t=1}^{5} \frac{t(t+1)C_t}{(1+0.18)^t} = \frac{1}{81.24}\left[\frac{1\times 2\times 12}{(1+0.18)} + \frac{2\times 3\times 12}{(1+0.18)^2} + ... + \frac{5\times 6\times 112}{(1+0.18)^5}\right]$$
$$= 10.78\ years^2$$

$$P_{25\%} = \sum_{t=1}^{5} \frac{C_t}{(1+0.25)^t} = \frac{12}{(1+0.25)} + \frac{12}{(1+0.25)^2} + ... + \frac{112}{(1+0.25)^5} = 65.04$$

$$Actual\ \%\ Price\ Change = \frac{65.04 - 81.24}{81.24} = -19.94\%$$

$$\%\ Price\ Change\ Using\ Duration = -D\frac{dy}{1+y} = -3.93 \times \frac{0.25-0.18}{1+0.18} = -23.28\%$$

$$\%\ Price\ Change\ Using\ Duration\ \&\ Convexity = -D\frac{dy}{1+y} + C\left(\frac{dy}{1+y}\right)^2 = -19.49\%$$

| | | | | | | |
|---|---|---|---|---|---|---|
| TIMELINE | 0 | 1 | 2 | 3 | 4 | 5 |
| YTM | | 0.18 | 0.18 | 0.18 | 0.18 | 0.18 |
| DISC FACTOR | | 0.84745763 | 0.71818 | 0.60863 | 0.5157889 | 0.4371 |
| CASH FLOW | | 12 | 12 | 12 | 12 | 112 |
| DCF | 81.23697 | 10.1694915 | 8.61821 | 7.30357 | 6.1894665 | 48.956 |
| tC(t) | | 12 | 24 | 36 | 48 | 560 |
| DISC tC(t) | 318.8557 | 10.1694915 | 17.2364 | 21.9107 | 24.757866 | 244.78 |
| DURATION | 3.925007 | | | | | |
| t(t+1)C(t) | | 24 | 72 | 144 | 240 | 3360 |
| DISC t(t+1)C(t) | 1752.167 | 20.3389831 | 51.7093 | 87.6428 | 123.78933 | 1468.7 |
| CONVEXITY | 10.7843 | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| ACTUAL REVISED PRICE | | | | | | |
| TIMELINE | 0 | 1 | 2 | 3 | 4 | 5 |
| YTM | | 0.25 | 0.25 | 0.25 | 0.25 | 0.25 |
| DISC FACTOR | | 0.8 | 0.64 | 0.512 | 0.4096 | 0.3277 |
| CASH FLOW | | 12 | 12 | 12 | 12 | 112 |
| DCF | 65.03936 | 9.6 | 7.68 | 6.144 | 4.9152 | 36.7 |
| ORIGINAL PRICE | 81.23697 | | | | | |
| ACTUAL % CHANGE | -0.199387 | | | | | |

| | |
|---|---|
| GIVEN CHANGE IN YTM | 0.07 |
| % CHANGE IN PRICE USING DURATION | -0.232839 |
| CONVEXITY CORRECTION | 0.037951 |
| NET % CHANGE IN PRICE | -0.194888 |

Let us look at this example. You have a 12% coupon bond with a YTM of 18% and 5 years remaining to maturity. The original price of the bond works out to 81.24. This is the price that gives a YTM of 18% with a coupon of 12%. The duration comes out to be 3.93 years and the convexity 10.78 years$^2$.

Now, we work out the actual price change. For the new price, we use the YTM of 25%. The coupon rate remains the same. This price at YTM of 25% is 65.04 so that the actual % price change is 19.94%. This is the correct price change. This is the price change that has been calculated from the fundamentals.

Now, if we use only duration, the % price change works out to 23.28%. Compare this with 19.94% which is the correct one. Now, the convexity correction turns out to be 3.79% and, therefore, combining the two (duration and convexity), the net % price change works out 19.49% which is far more close to the actual price change of 19.94%, than the figure that we have using duration alone.

So that is the utility of convexity. Notwithstanding the fact that in this case the figures are significantly magnified because of the massive change in the YTM that I have assumed, there can be situations where the mere use of duration for approximating the price change may not be adequate if you have a portfolio that has a significant steepness, significant convexity embedded in it.