Financial Derivatives & Risk Management
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Lecture 19: Spot Interest Rates & YTM

Before I take up a detailed explanation of the mechanics of interest rate futures, bit is necessary to present certain pre-requisites in relation to interest rates and interest rate risk. We start with the concept of "spot rates"

Spot rates

Spot rate, S_{0T}, of a given maturity t=T is the interest rate at which at which an investor can make a deposit as of today for the said maturity i.e. if the deposit is made today, the interest rate that applies to the deposit is called the spot rate. In a sense, it is the current rate. Suppose an investor makes an investment for 6 months starting today, the rate that he gets is called the 6-month spot rate. The defining characteristic is that the cash-outflow on account of the deposit occurs today.

Technically, the spot rate for a given maturity, is given by the yield to maturity onzero coupon intruments of the same maturity.

Zero coupon bonds are those bonds that make No coupon payments during their lifetime. There is only one payment back to the investor and that is on the date of maturity of the bond. Thus, these bonds pay only one cash flow to the investor. Hence,

- Spot interest rates are the YTMs on bonds that pay only one cash flow to the investor.
- Spot rates are usually calculated and quoted for 6 monthly intervals and then annualized by doubling the 6-monthly rate.

For example, if I want the 6-month spot rate, it shall be equal to the Yyield-to-maturity of a zero coupon bond, which is trading in the market with the maturity of 6 months. Similarly, if I want spot rate for a one year deposit, I work out the yield to maturity of a zero coupon bond with the maturity of one year. The YTM can be worked out on the basis of the price at which these instruments are trading in the market.

Usually spot rates are worked out for 6 monthly intervals rather than annual intervals and then annualized for the purpose of per annum quotations.

We, now, take up the concept of yield to maturity.

Yield to maturity

Yield to maturity (YTM) is that discount rate which equates the present value of all future cash flows attributable to a particular security to its current price i.e. it is that discount rate at which the present value of all the cash flows emerging from that fixed income security are equal to the current price at which the bond is traded in the market. Thus,

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

Thus, since the zero coupon bond has only one cash flow and that single cash flow occurs at maturity i.e. t=T, the above formula gets simplified in the case of zero coupon bonds to:

$$P_0 = \frac{C_T}{(1+y)^T}$$

But, by definition of spot rate: $P_0 (1+S_{0T})^T = C_T$ *so that* $S_{0T} = y$

Therefore, we have y is equal to S_{0T}. S_{0T} is today's spot rate for an investment with maturity at t=T i.e. it is the rate that an investor gets if he makes a deposit today for a maturity of period T.

We take up an example to illustrate the above concepts:

EXAMPLE

Consider three bonds A, B and C. Face value of each bond is 1,000 and maturity is two years.

- Bond A is a zero coupon bond redeemable at face value
- Bond B is a par bond with annual coupons.
- Bond C is an annuity with two equal payments at the end of each of two years, quoting at par.
- The annual spot rates are S_{01} = 10% and S_{02}=15%

Calculate the YTM of each bond.

Solution

These are 3 bonds A, B, and C. Face value of each bond is 1000. Bond A is a zero coupon bond that redeems at face value. It is trading at a discount and it is going to be redeemed at 1000 at the end of two years.

Bond B is a par bond, par bond means it is quoting at its face value. So, the present market price of the bond B is 1000 and it would be redeemed at the end of two years at 1000. It pays annual coupons at a certain rate, say *c*. The coupons are paid annually.

Bond C is an annuity, annuity means it will give you two equal payments at the end of year one, and at the end of year two. It is also quoting at par i.e. at 1000. In this case, the redemption will not be at 1000. The redemption will be segregated into two parts, one at the end of the first year and the second at the end of second, in such a way that the total cash flow (that is redemption value plus interest) at the end of the first year and the total cash flow (that is redemption value plus interest) at the end of the second year are equal. Spot interest rates are given for one year and two year maturities. You have to work out the YTM of the bonds.

Bond A

The spot rate is given us 10 percent for the maturity of one year and 15 percent for maturity of two years. Discount factors has been worked out by using annual compounding. So, they are $(1.10)^{-1}$ and $(1.15)^{-2}$.

Bond A gives a cash flow of zero at the end of the first year, but it gives you a cash flow of 1000 at the end of the second year (it is redeemed at face value).

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+S_{0t})^t} = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \text{ so that since A has only on cash-flow}$$
$$\text{of } 1000 \text{ at } t=2 \text{ and } S_{02}=15\%, \text{ we have } \frac{1000}{(1+0.15)^2} = \frac{1000}{(1+y)^2} \Rightarrow y = 15\%$$

So its current market price is obtained by discounting just this one cash flow at the end of second year since there is no cash flow at the end of first year. YTM works out to 15 percent which is interestingly the two years spot rate. This is because the entire cash flow is taking place at the end of the second year.

Bond B

Bond B is quoting at a par value i.e. it is presently quoting at a market price of 1000. Hence, in this case, because it is a level coupon bond i.e. with constant coupon rate over its life, its YTM will be equal to its coupon rate. This can also be explicitly verified by calculating the YTM. We have:

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+S_{0t})^t} = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \text{ so that since B is quoting at par}$$
$$P_0 = 1000. \text{ Also } C_1 = 1000c,\ C_2 = 1000c + 1000,\ S_{01} = 10\%,\ S_{02} = 15\%$$
$$\text{where } c \text{ is the coupon rate, we have}$$
$$1000 = \frac{1000c}{(1+0.10)} + \frac{1000c+1000}{(1+0.15)^2} \Rightarrow c = 14.644\%$$
$$\text{Since the bond is quoting at par } y = c = 14.644\%$$
$$\text{Alternatively, we can calculate } y \text{ by putting the value of } c \text{ above}$$
$$1000 = \frac{146.44}{(1+y)} + \frac{1146.44}{(1+y)^2} \Rightarrow y = 14.644\%$$

Bond C

In the case of C, we have an annuity bond so it is paying the same amount of money 600.516 at the end of the first year and at the end of second year. This figure of 600.516 is arrived at by noting that the bond is quoted at par i.e. at its face value of 1000. We have:

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+S_{0t})^t} = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$
$$\text{Here the cash flows in both years is the same, but the bond is quoting at par.}$$
$$\text{Hence, } P_0 = 1000. \text{ Also } C_1 = C_2 = A(say),\ S_{01} = 10\%,\ S_{02} = 15\%$$
$$1000 = \frac{A}{(1+0.10)} + \frac{A}{(1+0.15)^2} \Rightarrow A = 600.516$$
$$\text{We can calculate } y \text{ by putting the value of } A \text{ above}$$
$$1000 = \frac{600.516}{(1+y)} + \frac{600.516}{(1+y)^2} \Rightarrow y = 13.1324\%$$
$$\text{by solving the quadratic equation or using MSExcel.}$$

TIMELINE	YTM	0	1	2
SPOT RATE (P.A.)			10%	15%
DISCOUNT FACTOR			0.90909	0.75614
BOND A	0.15	-756.14367	0	1000
BOND B	0.14644	-1000.0006	146.44	1146.44
BOND C	0.131324	-1000	600.516	600.516

<u>YTM as an average of the spectrum of interest rates i.e. term structure</u>

Interestingly, YTM of all the three bonds A, B & C lie between 10 percent and 15 percent i.e. between the minimum and maximum spot rates. Further, as the intermediate cash flow increases, the YTM tilt towards the shorter term interest rate. This is because the weightage given to the intermediary cash flow increases. For zero coupon bond A, the weightage is zero. For B, it is 146.44, and for C it is 600.516 and the corresponding YTMs are respectively 15%, 14.644% and 13.1324% shifting towards the one year rate of 10% from the two year rate of 15%. So, in some sense this ratio of cash-flows in the various years contributes as weightages to the YTM tilting towards the first year rates from the second year. Consider the following expression:

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{\left(1+S_{0t}\right)^t} = \sum_{t=1}^{T} \frac{C_t}{\left(1+y\right)^t}$$

When an investor invests in a fixed income security, he gets periodical interest payment (coupons) at various points in time during the life of the bond followed by refund (redemption) of principal. Let us say it is an annual coupon bond. So he gets coupons at the end of each year up to the life of the bond at which point he gets the redemption money and the final coupon payment. Let us call the cash-flow at $t=t$ as C_t.

Now, since these cash-flows are occurring at different points in time, they need to be discounted at the rates appropriate to these maturities i.e. the cash-flow at the end of the first year needs to be discounted at the on-year spot rate S_{01} and so on. This issue assumes significance because of the phenomenon of term structure of interest rates i.e. that spot interest rates are functions of maturities. In other words, the spot rate applicable to a deposit of one year (say) may be slightly different from the spot rate applicable for a two-year deposit and so on.

It is very common knowledge in the case of fixed deposits with banks, banks generally have deposit rates varying with the maturities of fixed deposits. The same holds in the case of market interest rates as well.

The interest rates are function of maturities of the deposits. So, there is a spectrum of interest rates corresponding to different maturities, and that is what is called the term structure.

Therefore, because an investment in a bond yields cash flows at different points in time, it becomes necessary to account for this phenomena of term structure and use discount rates calculated using interest rates corresponding to the maturities equal to the timings of the cashflows. For example, in the case of cash flow in the end of the first year, you will require a rate which is the one-year spot rate S_{01}. A cash flow occurring at the end of the second year has to be discounted at the rate of S_{02} and so on.

So, we arrive at the current price of the bond as the aggregate present value of all future cash flows discounted at the respective spot rates.

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{\left(1+S_{0t}\right)^t}$$

But, by definition, but by definition of YTM:

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{\left(1+y\right)^t}$$

From these two equations, it is clearly seen that yield to maturity is that single rate, which discounts all future cash flows and equates them to the current market price. Thus, we have two expressions for the price, one based on the spectrum or the term structure of the interest rates corresponding to the timing of cash flows and the second on that one rate, one single rate, equalling the aggregate of discounted cash flows to equal the current price.

It means that this YTM (y) captures the effect, in some averaging sense (y is one number) of a multitude of numbers representing the various interest rates corresponding to different maturities i.e. the term structure. In other words, this y in a single figure captures the essence of the term structure of interest rates into one single figure. And that is why it is sometimes interpreted as an average, or a proxy of the market interest rates.

<u>Why YTM?</u>

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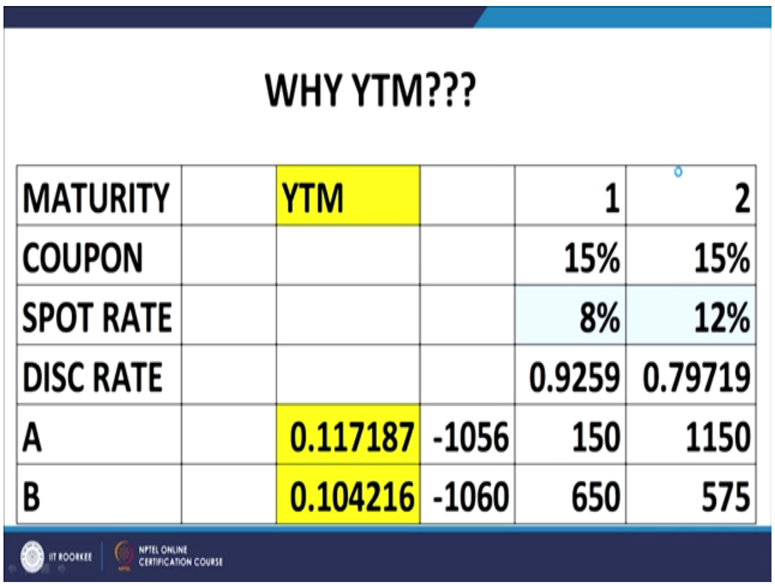

MATURITY		YTM		1	2
COUPON				15%	15%
SPOT RATE				8%	12%
DISC RATE				0.9259	0.79719
A		0.117187	-1056	150	1150
B		0.104216	-1060	650	575

Now the next question that we need to address is - why do we need YTM? For this purpose, we look at the above example. We are given a two 2-year bonds A & B both with a face value of 1,000. The coupon rate is 15% for both bonds but Bond A is redeemed in one instalment at face value at the end of second year while bond B is redeemed in two equal instalments of 500 each at the end of first year and second year. Thus, the cash flows of bond A will be 150 in Year 1 and

1,150 in Year 2 while for Bond B they will be 650 in year 1 and 575 in Year 2 since in Year 2 bond B will pay interest only on the outstanding amount of 500. Spot rates are taken as 8% and 12%.

Interestingly although the coupon rates as well as the spot rates are the same for both the bonds, the YTMs are not the same. They are 11.72% for A and 10.42% for B

$$Bond\ \ A: P_0 = \sum_{t=1}^{T} \frac{C_t}{\left(1+S_{0t}\right)^t} = \sum_{t=1}^{T} \frac{C_t}{\left(1+y\right)^t}$$
$$Also\ C_1 = 150,\ C_2 = 150+1000,\ S_{01} = 8\%,\ S_{02} = 12\%$$
$$P_0 = \frac{150}{\left(1+0.08\right)} + \frac{150+1000}{\left(1+0.12\right)^2} \Rightarrow P_0 = 1056$$
$$We\ can\ calculate\ \ y\ by\ \ putting\ \ the\ value\ of\ P_0\ \ above$$
$$1056 = \frac{150}{\left(1+y\right)} + \frac{1150}{\left(1+y\right)^2} \Rightarrow y = 11.72\%$$

$$Bond\ \ B: P_0 = \sum_{t=1}^{T} \frac{C_t}{\left(1+S_{0t}\right)^t} = \sum_{t=1}^{T} \frac{C_t}{\left(1+y\right)^t}$$
$$Also\ C_1 = 150+500,\ C_2 = 75+500,\ S_{01} = 8\%,\ S_{02} = 12\%$$
$$P_0 = \frac{150+500}{\left(1+0.08\right)} + \frac{75+500}{\left(1+0.12\right)^2} \Rightarrow P_0 = 1060$$
$$We\ can\ calculate\ \ y\ by\ \ putting\ \ the\ value\ of\ P_0\ \ above$$
$$1060 = \frac{650}{\left(1+y\right)} + \frac{575}{\left(1+y\right)^2} \Rightarrow y = 10.42\%$$

Now the question that YTM attempts to answer here is, which bond is better.

If we do not consider the YTM, it is very difficult to make a judgement of which of the two bonds is better, or which of the two bonds constitutes a superior investment. So YTM in this context comes to the rescue.

Spot rates vs YTM

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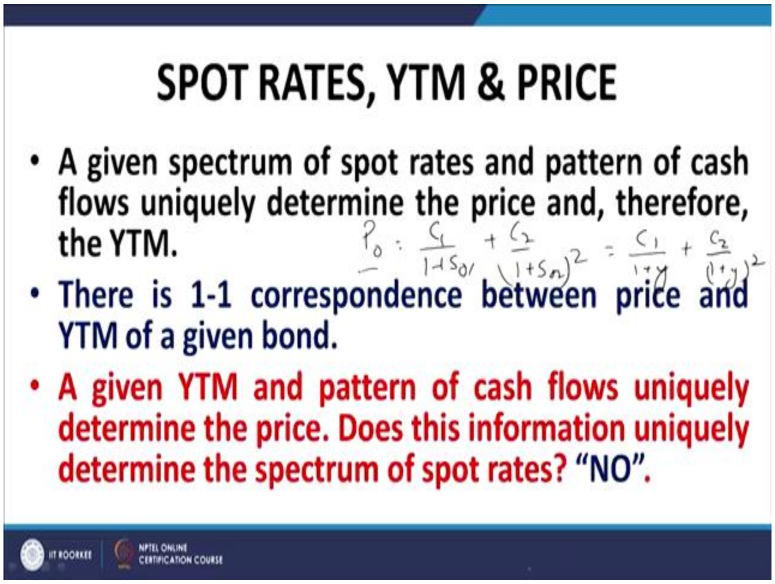

From the price equation:

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+S_{0t})^t}$$

it is quite apparent that given a certain cashflow pattern, and the corresponding spectrum of spot rates, we can uniquely determine the price of a bond. Further, knowing the price of the bond, the YTM is uniquely determined by:

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

Thus, for a particular bond there is one to one correspondence between the price and the YTM. It follows that a cashflow structure consisting of amounts and timings of the cash-flows and the corresponding spectrum of spot rates uniquely determine the YTM of the instrument.

However, given a price or given a YTM and a cash-flow structure i.e. a bond, one cannot uniquely determine the spectrum of spot rates. In other words, if one has a pattern of cash flows and the YTM, one can determine the price uniquely. That is fine. But one cannot uniquely determine the spot rate spectrum.

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MATURITY	YTM		1	2
SPOT RATE			0.08	0.12
DISC RATE			0.92593	0.79719
BOND PRICE	0.1042	-1060	650	575
NEW SPOT RATE		-1060	0.0829	0.11803

Let us see an example. We have one pair of spot rates as S_{01}: 8% and S_{02}: 12% and we considera bond with cashflows of 650 at the end of first year and 575 at the end of second year. The price of the bond is 1060 which translates to a YTM of 10.42%.

$$Bond\ \ B: P_0 = \sum_{t=1}^{T} \frac{C_t}{\left(1+S_{0t}\right)^t} = \sum_{t=1}^{T} \frac{C_t}{\left(1+y\right)^t}$$

$$Also\ C_1 = 150+500,\ C_2 = 75+500,\ S_{01} = 8\%,\ S_{02} = 12\%$$

$$P_0 = \frac{150+500}{\left(1+0.08\right)} + \frac{75+500}{\left(1+0.12\right)^2} \Rightarrow P_0 = 1060$$

$$We\ can\ calculate\ \ y\ by\ \ putting\ the\ value\ of\ P_0\ \ above$$

$$1060 = \frac{650}{\left(1+y\right)} + \frac{575}{\left(1+y\right)^2} \Rightarrow y = 10.42\%$$

Now, if instead of the above spot rates of 8% and 12% respectively, I use 8.29% as the one-year spot rate and 11.80% as the 2-year spot rate, I still end up with the same price and therefore, the same YTM. So two different pairs of interest rates yield the same YTM and price.

So, the net result is that while the YTM and the price are in one to one correspondence, YTM or price, do not uniquely determine this spectrum of interest rates.

$$P_0 = a_1\xi^1 + a_2\xi^2 = a_1\chi^1 + a_2\chi^2$$

$$\chi^1 = \xi^1 + \frac{a_2}{a_1}\left(\xi^2 - \chi^2\right)$$

Since there is only one eq with two unknowns, we have one dof that can be set arbitrarily. Hence, we have infinite no of solutions.

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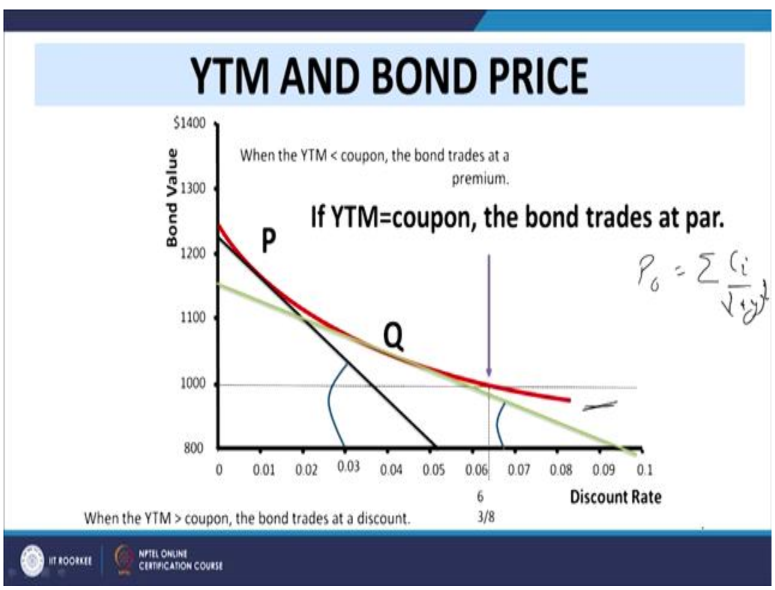

YTM & bond price

The YTM and price are inversely related. This is apparent from the YTM equation:

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

Furthermore,

(i) The value of a bond would be equal to its face value only when the coupon interest rate ` and YTM are equal.

(ii) The value of a bond would be higher (lower) than its face value when the coupon interest rate is higher (lower) than YTM.

(iii) The value of a bond would decrease (increase) as it approaches maturity, if it is a premium (discount) bond.

As mentioned earlier, YTM epitomizes the market rate in some sense. So, if YTM is less than the coupon, it implies that the average market rate for the life of the bond is less than the coupon rate of the bond. The coupon that the bond is giving is more than what the corresponding risk adjusted market rate demands. Thus, investors in this particular bond are getting more from that instrument compared to what the market equilibrium demands. This causes an increase in demand for this instrument with a corresponding rise in price, pushing down its return to the equilibrium levels.