Financial Derivatives & Risk Management
Professor J.P. Singh
Department of Management Studies
Indian Institute of Technology, Roorkee
Lecture 16: Capital Market Line & Security Market Line

We recall an example discussed earlier wherein we had considered a portfolio W of two securities A & B with $E(R_A)= E(R_B)=10\%$, $\sigma_A=\sigma_B=6\%$ and $\rho=0.50$. The portfolio W had $E(R_W)=10\%$ but $\sigma_W=5.20\%$. Thus, we had two securities e.g. A & W both with the same expected return but with different standalone risks. If the market prices standalone risk, then this situation cannot sustain itself in equilibrium due to arbitrage. Hence, an anomaly results. The solution to the anomaly lies in the outcome of the CAPM viz that market actually does not price standalone risk, rather it prices the systematic or beta component of the risk. However, if this argument is to hold, and beta is actually priced by the market, then beta must also scale according as expected return. In other words, it must be that the beta of a portfolio must be weighted average of the beta of its constituents, as is the case for expected returns. Let us see if it is so/. We have:

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1,\\ j\neq i}}^{N} X_iX_j\sigma_{ij} = \sum_{i=1}^{N} X_i^2\left(\beta_i^2\sigma_m^2+\sigma_{e_i}^2\right)+\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} X_iX_j\beta_i\beta_j\sigma_m^2$$

$$= \sum_{i=1}^{N} X_i^2\beta_i^2\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{e_i}^2 + \sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} X_iX_j\beta_i\beta_j\sigma_m^2$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} X_iX_j\beta_i\beta_j\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{e_i}^2 = \left(\sum_{i=1}^{N} X_i\beta_i\right)\left(\sum_{j=1}^{N} X_j\beta_j\right)\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{e_i}^2 = \beta_P^2\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{e_i}^2$$

Now, if we look at the term $\beta_P^2\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{e_i}^2$ for the portfolio with that for the individual security $\beta_i^2\sigma_m^2+\sigma_{e_i}^2$ it is apparent that β_P identifies with the security's β_I and that the term $\beta_P^2\sigma_m^2$ is the systematic risk of the portfolio. It follows, then, that the expression $\sum_{i=1}^{N} X_i\beta_i$ is the appropriate expression for the beta of the portfolio. Thus, the beta of the portfolio is the weighted beta of its constituents, as required by the market equilibrium characteristics.

In other words, just at the expected return of a portfolio of a combination of securities is the weighted average of expected return of its constituents, beta of a portfolio is also the weighted average beta of its constituents. $\sum_{i=1}^{N} X_i^2\sigma_{e_i}^2$ is the unsystematic risk of the portfolio.

Thus, the portfolio beta is the weighted average beta of the constituents. This makes this risk measure compatible with the return since returns also scale in the same manner.

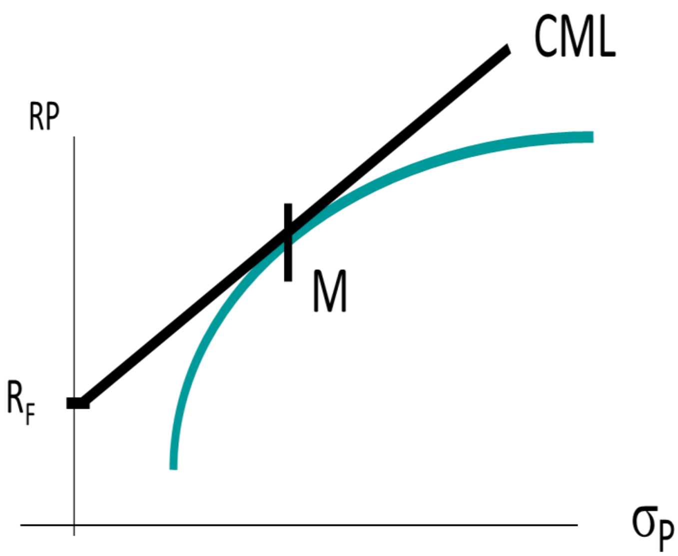
Capital market line (CML) & Security market line (SML)

The capital market line is the join of the risk free security and the market portfolio in the risk-return (σ, E(R)) space.

(i) Now, in the CAPM model, the optimal investment portfolio of every investor consists of the risky asset which coincides with the market portfolio (which is also the highest Sharp ratio portfolio) together with riskfree lending or borrowing.

(ii) We also know that that the portfolio possibilities curve of a risky asset and the riskfree asset is the straight line joining the two assets in the $(\sigma, E(R))$ space.

It, therefore, follows from (i) & (ii) that the portfolios of all the investors in the market (in the CAPM framework) must necessarily lie along the straight line joining the market portfolio & the riskfree asset in $(\sigma, E(R))$ space. But this is precisely the capital market line.



$$R_p = R_f + \frac{R_m - R_f}{\sigma_m}\sigma_p$$

The CML constitutes the efficient frontier in the CAPM model. Every portfolio lying on the CML is efficient. This is easily seen. Consider any arbitrary portfolio P on the CML. It consists of a combination of the market portfolio M and the riskfree asset F. Neither, the market portfolio nor the riskfree asset has any unsystematic risk. It follows that the unsystematic risk of P (given by $\sum_{i=1}^{N} X_i^2 \sigma_{e_i}^2$) must also vanish.

Portfolios along the CML are perfectly correlated with the market portfolio. We have, for an arbitrary portfolio P on the CML, $\sigma_P^2 = E\left[R_P - E\left(R_P\right)\right]^2$

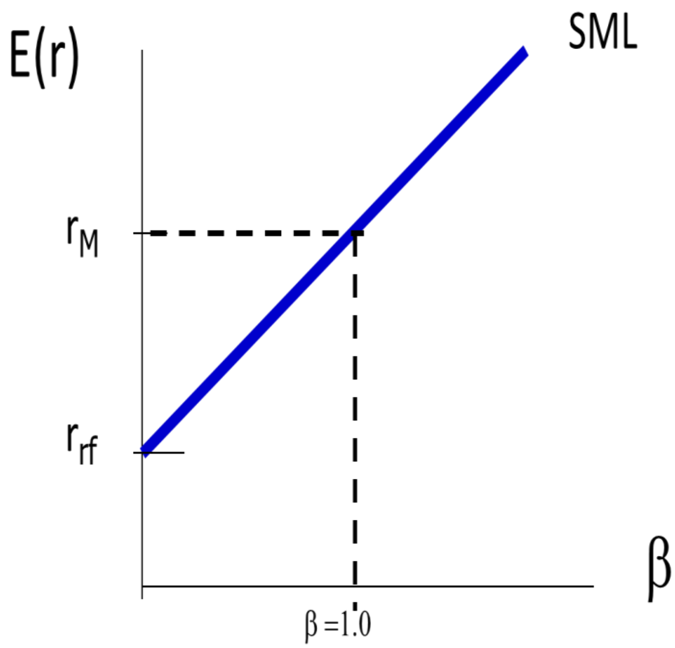
$= E\left[X_m R_m + X_F R_F - E\left(X_m R_m + X_F R_F\right)\right]^2 = E\left[X_m R_m - E\left(X_m R_m\right)\right]^2 = X_m^2 \sigma_m^2$

$$\sigma_{Pm} = E\left\{\left[R_P - E\left(R_P\right)\right]\left[R_m - E\left(R_m\right)\right]\right\}$$
$$= E\left\{\left[X_m R_m + X_F R_F - E\left(X_m R_m + X_F R_F\right)\right]\left[R_m - E\left(R_m\right)\right]\right\}$$
$$= E\left\{\left[X_m R_m - E\left(X_m R_m\right)\right]\left[R_m - E\left(R_m\right)\right]\right\} = X_m \sigma_m^2 \text{ so that } \rho_{Pm} = \frac{\sigma_{Pm}}{\sigma_P \sigma_m} = 1 .$$

Thus, we can say that only those portfolios are efficient under the CAPM model that are perfectly correlated with the market portfolio. This is a very important conclusion.

Security market line

The SML represents the CAPM in (β,E(R)) space. The CAPM gives us: $E(R_S)=R_F+\beta(E(R_M)-R_F)$. The plot of $E(R_S)$ against β_S is a straight line called the Security Market Line. Like the CML, SML also represents a risk-return relationship. However, in the case of SML, the measure of risk is β, the systematic risk of the security while in the CML it is the total risk σ. The SML is a plot in (β,E(R)) space.

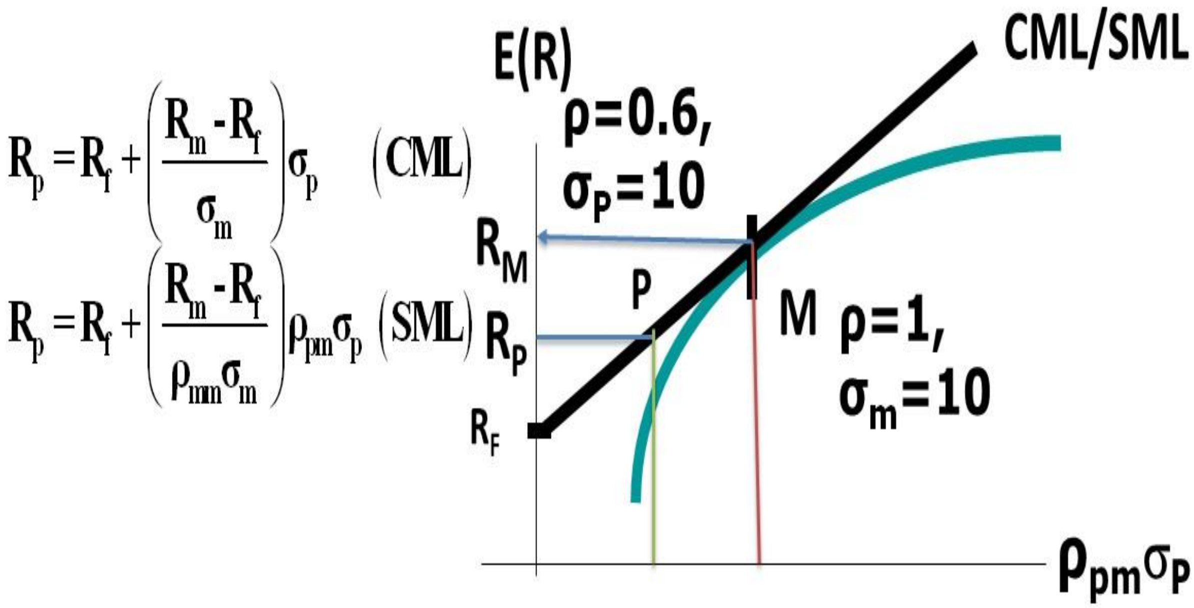


$$R_i = R_f + \frac{R_m - R_f}{\sigma_m}\rho\sigma_i = R_f + \beta_i\left(R_m - R_f\right)$$

Now, the CAPM risk-return relationship holds for all securities & portfolios irrespective of whether they are efficient or not. Therefore, unlike the CML which plots only the efficient portfolios i.e. those with zero unsystematic risk, the SNL plots all securities that are being traded in the market. But it relates only the systematic risk to their respective prices.

To further examine the relationship between CML & SML, we write:

Now, β is the regression coefficient of $E(R_S)$ on $E(R_m)$. Hence we can write it as: $\beta = \rho \frac{\sigma_S}{\sigma_m}$. Hence we can write the SML as $R_P = R_F + \left(\frac{R_m - R_F}{\rho_{mm}\sigma_m} \right) \rho_{Pm}\sigma_P$ since $\rho_{mm}=1$. We compare this eq with that of the CML $R_P = R_F + \left(\frac{R_m - R_F}{\sigma_m} \right) \sigma_P$. We find that the SML relationship (CAPM) can be represented in (ρσ,E(R)) space instead of (β,E(R)) space. The measure of risk in this space is total risk scaled by the correlation between market & security returns. Indeed, ρσ is an equivalent measure of systematic risk of a security since $\rho\sigma_i=\beta_i\sigma_m$. A very interesting observation is that in this space the slope of the SML $\left(\frac{R_m - R_F}{\sigma_m} \right)$ and the Y-intercept R_F are both precisely the same as those of the CML in this space. It follows that in the (ρσ,E(R)) space, the SML & CML coincide. However, the change in variable from β to ρσ needs to be handled with care. Suppose we are given R_F=5%, σ_M =10%, $E(R_M)$=20%, then for an efficient portfolio P with σ_P =20% and ρ=1 (since the portfolio is efficient), we get $E(R_P)$=35%. However, if the portfolio P is inefficient and the correlation between market & security returns is 0.50, then $E(R_P)$=20%. Thus, in this space the return that gets priced by the market is measured by ρσ instead of β. Systematic risk is measured by ρσ.



As another example, we consider two securities, M & P. Both of them have a standard deviation of 10%. M is the market portfolio and thus has ρ=1, whereas we assume ρ_{Pm}=0.60. If we plot these securities along the SML in (ρσ,E(R)) space, we get the points M & P respectively in the above figure. We also find the corresponding expected returns as R_M & R_P. Clearly the expected return on P is scaled down by its correlation with the market. The expected return that the market is willing to give to P is scaled down from R_M to R_P despite the two securities having the same total risk. This is happening because P is not perfectly correlated to the market portfolio M and hence, is not efficient and therefore carries an element of unsystematic risk. But the market does not attribute any expected return to this unsystematic risk. In fact, ρ_{Pm} happens to be 0.60 showing that only 60% of the total risk is what is being priced by the market.

In this framework the CML & SML are represented by the same line, but the risk measure gets changed to $\rho\sigma$. Since for all efficient portfolios $\rho=1$, we can use σ for these portfolios as the risk measure and we have the CML. However, for non-efficient portfolios we need to scale the total risk by the correlation with the market to arrive at the priced risk and then use the CML with that scaled measure of risk as the independent variable.

Example 1

A portfolio manager has maintained an actively managed portfolio with a beta of 0.2. During the last year, the risk-free rate was 5% and equities performed very badly providing a return of -30% overall. The portfolio manager produced a return of -10% and claims that in the circumstances it was a good performance. Discuss this claim.

Solution

Given $\beta=0.20$, $R_f=0.05$, $R_m=-0.30$. Using CAPM, $R_i= 0.05+0.20*(-0.30-0.05)= -0.02$
Actual $R_i=-0.10$, which clearly shows that the portfolio manager performed worse than a purely passive strategy

Example 2

The coordinates of two securities A & B in $(\sigma,E(R))$ space are respectively (8,12) and (12,24) with respective betas 1.2 and 2.0. A portfolio is proposed to be constituted comprising of these securities A & B in the ratio 3:1 (in terms of the amount of investment). The variance of the market portfolio is $25(\%)^2$. Assuming that the CAPM holds, calculate the systematic and unsystematic risk of the portfolio so constituted, expressed in $(\%)^2$.

Solution

				A	B	
EXP RETURN				12	24	
SD				8	12	
TOTAL RISK (VAR)				64	144	
BETA				1.2	2	
MARKET VAR			25			
SYS RISK				36	100	
UNSYS RISK				28	44	
COMPOSITION (Xi)				0.75	0.25	
(Xi)^2				0.5625	0.0625	
UNSYS RISK OF PORTFOLIO						18.5

Index futures will be covered in the next note for continuity.