Financial Derivatives & Risk Management
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Lecture 14: Capital Asset Pricing Model

The Capital Asset Pricing Model

In the Markowitz mean-variance optimization framework we arrived at the optimal portfolio by maximizing the expression $Z = \frac{E(R_P) - R_F}{\sigma_P}$ i.e. excess expected return per unit standard deviation. This ratio is called the Sharpe ratio. The cardinal assumptions of the Markowitz model were:

(i) Expected return is a function of risk.
(ii) Risk is measurable by standard deviation.

Now, we move from Markowitz to CAPM. Now Markowitz framework looks at the investment problem from the individual investor's perspective. How the investor achieves the optimal portfolio is attempted to be explained by the Markowitz framework.

William Sharpe addressed the investment problem from a different perspective. He looked at the problem from the market's perspective. He attempted to answer the question:

"What are the implications for asset prices if everyone adopts the Markowitz framework? In equilibrium, all assets must be held by someone. For the market to be in equilibrium, the expected return of each asset must be such that investors collectively decide to hold exactly the supply of shares of the asset."

He assumed that all the investors in the market do the optimization exercise as per Markowitz mean variance optimization framework. In that situation, if the market is to be in equilibrium, then the prices of various securities should align themselves in such a way that the supply and demand of every security must be equal. The supply must equal the demand for each security. If that be so, then how does the market, as a collective entity, go about pricing of securities? Since prices determine returns, the question could be posed as: "What was the functional relationship between risk and return as priced by the market?" or equivalently,

(i) "Was standalone risk measured by standard deviation of a security an appropriate measure of risk, as priced by the market?",
(ii) "Does market price standalone risk?"

In other words, if an investor takes a higher standalone risk, will he "expect" a higher return? Does market behave that way? Secondly, whether market risk was measurable by standalone risk? These were the questions Sharpe tried to answer.

Standalone risk not appropriate

Let us try to understand why this question arises. Assume two securities A & B with equal standard deviation, which is a measure of standalone risk, as $\sigma_A=\sigma_B=6\%$. Both the securities are priced in such a way that the expected returns $E(R_A)=E(R_B)=10\%$. So, both securities have the same standalone risk and are priced to the same expected return.

Let the correlation between them be $\rho=0.50$. Suppose we constitute a portfolio W comprising of equal weights of A & B. Then, clearly $E(R_W)=10\%$. But the standalone risk (standard deviation) works out to $\sigma_W=5.20\%$.

If market priced risk by the standalone risk (measured by standard deviation), then the portfolio W being less risky should have been priced with a lower expected return contrary to the above computations. Thus, we arrive at a contradiction. We have two securities say A & W that have the same expected return but the standalone risk of A exceeds the standalone risk of the other W. Putting it another way, W has a lower standalone risk than A but prices the two equivalently in terms of expected return. IT follows that the market does not price securities in line with their standalone risk.

Improving the Sharpe ratio & optimality

Recall that in the Markowitz mean-variance optimization framework, we optimized our portfolio problem by maximizing the Sharpe ratio viz $Z=\frac{E(R_P)-R_F}{\sigma_P}$ i.e. excess expected return per unit standard deviation.

When an investor reaches the point at which the Sharpe ratio is maximum, the portfolio is the optimum portfolio. Now, if that is the optimization criterion, then an investor would continue adding a security to his portfolio upto the point that the adding of that security to his portfolio improves his Sharpe ratio.

When he does not achieve any further improvement in Sharpe ratio by adding any new security to his portfolio, he achieves the optimal portfolio. Hence, optimization implies obtaining the condition under which an investor's portfolio's Sharpe ratio improves. In other words, the Sharpe ratio of the portfolio obtained by adding a new security to the existing portfolio is better than the the Sharpe ratio of existing portfolio. If the addition of a security S to a portfolio P increases the Sharpe ratio of the revised portfolio T, then it should be so added.

Consider an investment of one unit of money in a portfolio P. Let us assume that a small fraction in value (x) of a security S is added to P to get portfolio T. Then,

$$E(R_T)=\frac{1}{1+x}E(R_P)+\frac{x}{1+x}E(R_S) \text{ and}$$

$$\sigma_T=\left(\frac{1}{1+x}\right)\left(\sigma_P^2+x^2\sigma_S^2+2\rho x\sigma_P\sigma_S\right)^{1/2}\approx\left(\frac{1}{1+x}\right)\left(\sigma_P^2+2\rho x\sigma_P\sigma_S\right)^{1/2}\approx\sigma_P\left(\frac{1}{1+x}\right)\left(1+x\rho\frac{\sigma_S}{\sigma_P}\right)$$

In the above expression, we have taken first order approximations so that higher powers of x have been neglected. The binomial expansion has also been taken to first order $(1+x)^{1/2}\approx 1+\frac{1}{2}x$.

The Sharpe ratio after the addition of S changes to:

$$\Theta_T = \frac{E(R_T) - R_F}{\sigma_T} = \frac{E(R_P) + xE(R_S) - (1+x)R_F}{\sigma_P\left(1 + x\rho\frac{\sigma_S}{\sigma_P}\right)}$$

For $\Theta_T > \Theta_P$, we need $\frac{E(R_P) + xE(R_S) - (1+x)R_F}{\sigma_P\left(1 + x\rho\frac{\sigma_S}{\sigma_P}\right)} > \frac{E(R_P) - R_F}{\sigma_P}$ or

$$E(R_P) + xE(R_S) - (1+x)R_F > \left(1 + x\rho\frac{\sigma_S}{\sigma_P}\right)\left[E(R_P) - R_F\right] \text{ or}$$

$$E(R_S) - R_F > \rho\frac{\sigma_S}{\sigma_P}\left[E(R_P) - R_F\right] \text{ or } E(R_S) - R_F > \beta_S\left[E(R_P) - R_F\right]$$

At this point there are two important issues:

(i) We have, nowhere, mandated that x>0 i.e. that x must be positive or that the security S must be longed into the portfolio P to achieve T. The above analysis perfectly holds if the security S is shorted into P to form portfolio T i.e. x is negative. In other words, since this model does not restrict short sales, if the Sharpe ratio of a portfolio can be improved by shorting a security into it, it is also very much permissible and would ne necessary in the optimization; optimization criterion must be tested both with long and short positions in all the available securities. If the Sharpe ratio improves by shorting an available security, it is allowed and needs to be done for achieving optimality.

(ii) Since the number of securities in the market are finite, this optimization process will come to a finite end, at whch point no further addition will increase the Sharpe ratio. At this point, we will necessarily have $E(R_S) - R_F = \beta_S\left[E(R_P) - R_F\right]$ because the very existence of '>' would imply that further optimization can be done. Thus, at equilibrium we must have $E(R_S) - R_F = \beta_S\left[E(R_P) - R_F\right]$.

This is a very important equation, it leads us to the CAPM.

We make some radical assumptions which are embedded in CAPM viz.

(i) All investors will determine the same highest Sharpe Ratio portfolio of risky assets.

(ii) Depending on their risk tolerance, each investor will allocate a portion of wealth to this optimal portfolio and the remainder to risk-free lending or borrowing.

(iii) Investors will hold risky assets in the same relative proportions.

(iv) This proportion will, therefore, also constitute the market composition of risky assets.

(v) In equilibrium, therefore, the portfolio of risky assets with the highest Sharpe Ratio must be the market portfolio.

(vi) If the market portfolio has the highest attainable Sharpe Ratio, there is no way to obtain a higher Sharpe Ratio by holding more or less of any one asset.

Applying the portfolio improvement rule, it follows that the risk premium of each asset must satisfy $R_s=R_f+\beta_s (R_m-R_f)$.

The first assumption follows from the underlying that the inputs that go into the determination of the optimization process of each investor are assumed to be the same for all the investors in the market. Further, all investors do the portfolio optimization in the mean-variance framework of Markowitz. It therefore, follows that every investor will determine the same highest Sharpe ratio portfolio. Thus, the fact of a universal highest Sharpe ratio portfolio follows directly from the homogeneity of investors expectations of the investors about the inputs to the Markowitz optimization model and the use of the Markowitz model by all investors for portfolio optimization.

Stated equivalently, all the investors will hold the same risky portfolio i.e. the highest Sharpe ratio portfolio which happens to be the same for all investors. ***A consequence of this is that the market outstandings (in value) of all the risky assets will also be in the same proportion i.e. the proportion in which these risky assets feature respectively in the highest Sharpe ratio portfolio.***

It follows that the market portfolio will have exactly the same constitution as the highest Sharpe ratio portfolio.

Let us take an example, suppose there are two investors in the market A & B, and two securities X & Y. Let us assume A holds 50,000 worth of X and 100,000 worth of Y and B holds 10,000 woth of X and 20,000 worth of Y. Clearly, the market outstanding will be 60,000 worth of X and 120,000 worth of Y i.e. in the same ratio as the holdings of A & B.

Pertinent to mention here that the actual portfolio invested in by an investor shall not be only and entirely the highest Sharpe ratio portfolio. Rather it would be a combination of the highest Sharpe ratio portfolio together with riskfree lending or borrowing, as the case may be, depending of his risk tolerance. However, the risky constituent of this investor portfolio shall be provided by the universal highest Sharpe ratio portfolio. In other words, the market simply consists of two assets viz the risky highest Sharpe ratio market portfolio and the riskfree asset. Every investor holds some combination of the two, determined solely by his risk tolerance.

Coming back to the risky (highest Sharpe ratio) portfolio, if the highest Sharpe ratio portfolio coincides with the market portfolio, as explained above, there is no way to obtain a higher Sharpe Ratio by holding more or less of any one asset.

In view of the above, the portfolio P, in the above analysis which is the highest Sharpe ratio portfolio on optimization, also becomes the market portfolio M so that we have:

$E(R_S)=R_F+\beta_S[E(R_m)-R_F]$.

This is the CAPM equation.

<u>Regression, yet again!</u>

It is clearly seen from the CAPM equation $E(R_S)=R_F+\beta_S[E(R_m)-R_F]$ that it relates the expectation values of two variables R_S, R_m. Thus, it is clearly of the form of a regression equation. It is emphasized that this equation holds “on the average”. The given regression equation premises a systematic relationship between (R_m-R_F) which is termed as the risk premium and R_S. A particular measurement is unlikely to absolutely match the value predicted by this return and can, therefore, be represented by: $R_{S,i}=R_F+\beta_S[E(R_{m,i})-R_F]+e_i$ where e_i is a random error term that represents the factors not modelled by this regression equation but nevertheless having influence on the security’s returns. It is the unsystematic component of the return, the random return.

A consequence of this orthogonal partitioning of return into systematic & unsystematic components is that risk is also partitioned accordingly. Putting it the other way round, the return is partitioned into a component which is related to systematic risk, which is called risk premium and a random component associated with unsystematic risk.

This systematic risk is rewarded by the market by the risk premium and the unsystematic risk continues to be random and hence, unrewarded by market pricing.

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So, let us summarize:

(i) Standalone risk is measured by standard deviation. However, it is not an appropriate measure of market risk. Market does not price standalone risk.

(ii) Beta is a measure of systematic risk, which is priced by the market.

(iii) The total risk of the portfolio can be segregated into two orthogonal components, the systematic risk, which is the beta risk and the unsystematic risk, which is the random risk. Market does not price total risk. Market prices systematic risk.

(iv) Market does not reward investors for taking unsystematic risk.

CAPM: Alternative Derivation from the MVO Equations

The MVO eqs. are $\lambda \sum_{j-1}^{N} X_j \sigma_{jk} = \bar{R}_k - R_F$

Now, $\sigma_{jk} = E\left[\left(R_j - \bar{R}_j\right)\left(R_k - \bar{R}_k\right)\right]$ *so that*

$$\sum_{j-1}^{N} X_j \sigma_{jk} = \sum_{j-1}^{N} X_j E\left[\left(R_j - \bar{R}_j\right)\left(R_k - \bar{R}_k\right)\right] = E\left[\left(\sum_{j=1}^{N} R_j X_j - \sum_{i=1}^{N} \bar{R}_j X_j\right)\left(R_k - \bar{R}_k\right)\right]$$

Under the assumptions of CAPM (homogeneity of expectations) $\sum_{j=1}^{N} R_j X_j = R_M$

since each investor is faced with the same tangency portfolio

which thus becomes the market portfolio, so that $\sum_{j-1}^{N} X_j \sigma_{jk} = \sigma_{kM}$

For $k - M$*, we have* $\sum_{j-1}^{N} X_j \sigma_{jM} = \sigma_M^2$ *whence* $\lambda \sigma_M^2 = \bar{R}_k - R_F$

or $\lambda = \frac{\bar{R}_k - R_F}{\sigma_M^2}$ *giving* $\frac{\bar{R}_k - R_F}{\sigma_M^2} \sigma_{kM} = \bar{R}_k - R_F$