Financial Derivatives & Risk Management
Professor J.P Singh
Department of Management Studies
Indian Institute of Technology, Roorkee
Lecture 13: Mean Variance Portfolio Theory

Markowitz Mean Variance Portfolio Theory

The fundamental premise of the Markowitz model is that investment decisions are made on a two dimensional framework, risk and return. Risk is measured in terms of the standard deviation while for return we use the expected return i.e. the expected percentage change in price per unit time. Expected return is believed to be a function of standard deviation. This is the risk-return framework on which investments are analyzed.

Return and Variance of a Portfolio of Securities

The instantaneous, expected returns and variances of a portfolio of N securities with composition vector $\mathbf{X}=\{X_i, i=1,2,3,...,N\}$, $\sum_{i=1}^{N} X_i = 1$ are given respectively by:

$$R_P = \sum_{i=1}^{N} X_i R_i \tag{1}$$

$$E(R_P) = \sum_{i=1}^{N} X_i E(R_i) \tag{2}$$

$$\sigma_P^2 = E\left[R_P - E(R_P)\right]^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j \sigma_{ij} = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N} X_i X_j \sigma_{ij} = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + 2\sum_{i=1}^{N}\sum_{\substack{j=1 \\ i<j}}^{N} X_i X_j \sigma_{ij} \tag{3}$$

The Portfolio Possibilities Curve (PPC) for Two Security Portfolio

We define the portfolio possibilities curve (PPC) as the locus of a point in risk-return space that identifies an admissible portfolio. For a two security portfolio with composition vector $\mathbf{X}=\{X_1, 1-X_1\}$, we have, from eqs. (2), (3) with $\rho = \sigma_{12}\sigma_1^{-1}\sigma_2^{-1}$

$$E(R_P) = X_1 E(R_1) + (1-X_1) E(R_2) \tag{4}$$

$$\sigma_P^2 = X_1^2\sigma_1^2 + (1-X_1)^2 \sigma_2^2 + 2X_1(1-X_1)\rho\sigma_1\sigma_2 \tag{5}$$

Eliminating X_1 between eqs. (4) & (5), we obtain the equation for the PPC for the two security case as:

$$x^2 - y^2 \frac{\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)}{\left(R_1 - R_2\right)^2} + 2y \frac{\left[R_2\sigma_1^2 + R_1\sigma_2^2 - \left(R_1 + R_2\right)\rho\sigma_1\sigma_2\right]}{\left(R_1 - R_2\right)^2}$$

$$-\frac{\left(R_2^2\sigma_1^2+R_1^2\sigma_2^2-2R_1R_2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2}=0 \qquad (6)$$

where we have abbreviated $E\left(R_P\right)\equiv y,\sigma_P=x,E\left(R_1\right)\equiv R_1,E\left(R_2\right)\equiv R_2$.

Comparing eq. (6) with the general equation of a conic viz.

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad (7)$$

we obtain $a=1,h=0,b=-\dfrac{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2}$ whence $h^2-ab>0$ so that the PPC represents a hyperbola in shape. The equation of the PPC can be written as

$$x^2-by^2+2fy-c=0 \text{ or } \frac{x^2}{c-\dfrac{f^2}{b}}-\frac{\left(y\sqrt{b}-\dfrac{f}{\sqrt{b}}\right)^2}{c-\dfrac{f^2}{b}}=1 \qquad (8)$$

where

$$b=\frac{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2},\ f=\frac{\left[R_2\sigma_1^2+R_1\sigma_2^2-\left(R_1+R_2\right)\rho\sigma_1\sigma_2\right]}{\left(R_1-R_2\right)^2},c=\frac{\left(R_2^2\sigma_1^2+R_1^2\sigma_2^2-2R_1R_2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2} \qquad (9)$$

We have a curve representing the combinations of the two securities. Only those portfolios are possible such that the points corresponding to their standard deviation and expected return lie on this particular curve. Only those risk-return combinations can be achieved by using the given pair of securities, which lie on the particular curve. We cannot cover the entire risk-return space by using combinations of these two securities. Further, every point on this curve corresponds to a unique combination of the given two securities.

It is important at this point to note the following:

(a) $x\equiv\sigma$ represents standard deviation of a random variable (security returns) and hence, can never be negative;

(b) Assuming **<u>no short sales</u>**, the portfolio return $y\equiv R_P$ must necessarily lie between R_1 & R_2 so that no point of the PPC can lie outside the region bounded by the abscissa through R_1 & R_2;

(c) We must also have $-1\le\rho\le1$. Let us, now, examine these two extremal cases:
For perfectly correlated assets, $\rho=+1$, eq. (6) becomes

$$y=\frac{\left(R_1-R_2\right)}{\left(\sigma_1-\sigma_2\right)}x+\frac{\left(R_2\sigma_1-R_1\sigma_2\right)}{\left(\sigma_1-\sigma_2\right)} \tag{10}$$

which is a straight line with gradient $\frac{\left(R_1-R_2\right)}{\left(\sigma_1-\sigma_2\right)}$, intercept on the ordinate axis $\frac{\left(R_2\sigma_1-R_1\sigma_2\right)}{\left(\sigma_1-\sigma_2\right)}$ and passing through the points $A\left(\sigma_1,R_1\right)$ & $B\left(\sigma_2,R_2\right)$ representing the two securities in risk-return space. Hence, any portfolio of two perfectly correlated securities will lie on the straight line joining the two securities in risk-return space and the PPC, in this case, is the straight line joining these two points. The case of anti-correlated assets $(\rho=-1)$ is relatively more involved. The eq. of the PPC becomes

$$\left(R_1-R_2\right)x=\pm\left[\left(\sigma_1+\sigma_2\right)y-\left(R_1\sigma_2+R_2\sigma_1\right)\right] \tag{11}$$

Since x being standard deviation must necessarily be positive, the sign of the RHS is dictated by the sign of $\left(R_1-R_2\right)$ so that we shall have two scenarios and hence, a pair of straight lines

(i) in the region where $\operatorname{sgn}\left(R_1-R_2\right)=\operatorname{sgn}\left[\left(\sigma_1+\sigma_2\right)y-\left(R_1\sigma_2+R_2\sigma_1\right)\right]$ the positive sign outside the square bracket will hold and the equation of the PPC in this region will be

$$y=\frac{\left(R_1-R_2\right)}{\left(\sigma_1+\sigma_2\right)}x+\frac{\left(R_1\sigma_2+R_2\sigma_1\right)}{\left(\sigma_1+\sigma_2\right)} \tag{12}$$

(ii) in the region where $\operatorname{sgn}\left(R_1-R_2\right)\neq\operatorname{sgn}\left[\left(\sigma_1+\sigma_2\right)y-\left(R_1\sigma_2+R_2\sigma_1\right)\right]$, the negative sign outside the square bracket will hold and the equation of the PPC in this region will be

$$y=-\frac{\left(R_1-R_2\right)}{\left(\sigma_1+\sigma_2\right)}x+\frac{\left(R_1\sigma_2+R_2\sigma_1\right)}{\left(\sigma_1+\sigma_2\right)} \tag{13}$$

In fact, by an appropriate numbering of the two securities, we can always ensure that $\left(R_1-R_2\right)\geq 0$ whence eq. (12) will operate in the region where $y\geq\frac{\left(R_1\sigma_2+R_2\sigma_1\right)}{\left(\sigma_1+\sigma_2\right)}$ or equivalently $X_1\geq\frac{\sigma_2}{\left(\sigma_1+\sigma_2\right)}$ and eq. (13) in the region where $X_1\leq\frac{\sigma_2}{\left(\sigma_1+\sigma_2\right)}$. It is pertinent to mention that the straight lines (12) & (13) intersect each other and the ordinate axis at the point $F\left(0,\dfrac{(R_1\sigma_2+R_2\sigma_1)}{(\sigma_1+\sigma_2)}\right)$ which identifies the risk free rate of return. Further, eq. (12) is the join of the point F & A while (13) is the join of F & B so that for the entire range of values $0\le X_1\le 1$, the risk free ordinate $F\left(0,\dfrac{(R_1\sigma_2+R_2\sigma_1)}{(\sigma_1+\sigma_2)}\right)$ is unique. Needless to add, the return $R_F=\dfrac{(R_1\sigma_2+R_2\sigma_1)}{(\sigma_1+\sigma_2)}$ lies between R_1, R_2.

The fallout of the observations above is that the PPC shall be confined to the section of the hyperbola lying in the first quadrant between the lines given by eqs. (10), (12) & (13) that, incidentally form a triangle with the vertices $A(\sigma_1,R_1)$, $B(\sigma_2,R_2)$ and $F\left(0,\dfrac{(R_1\sigma_2+R_2\sigma_1)}{(\sigma_1+\sigma_2)}\right)$.

The exact shape of the hyperbola is parameterized by the correlation coefficient between the two given securities, ρ (Figure 1).

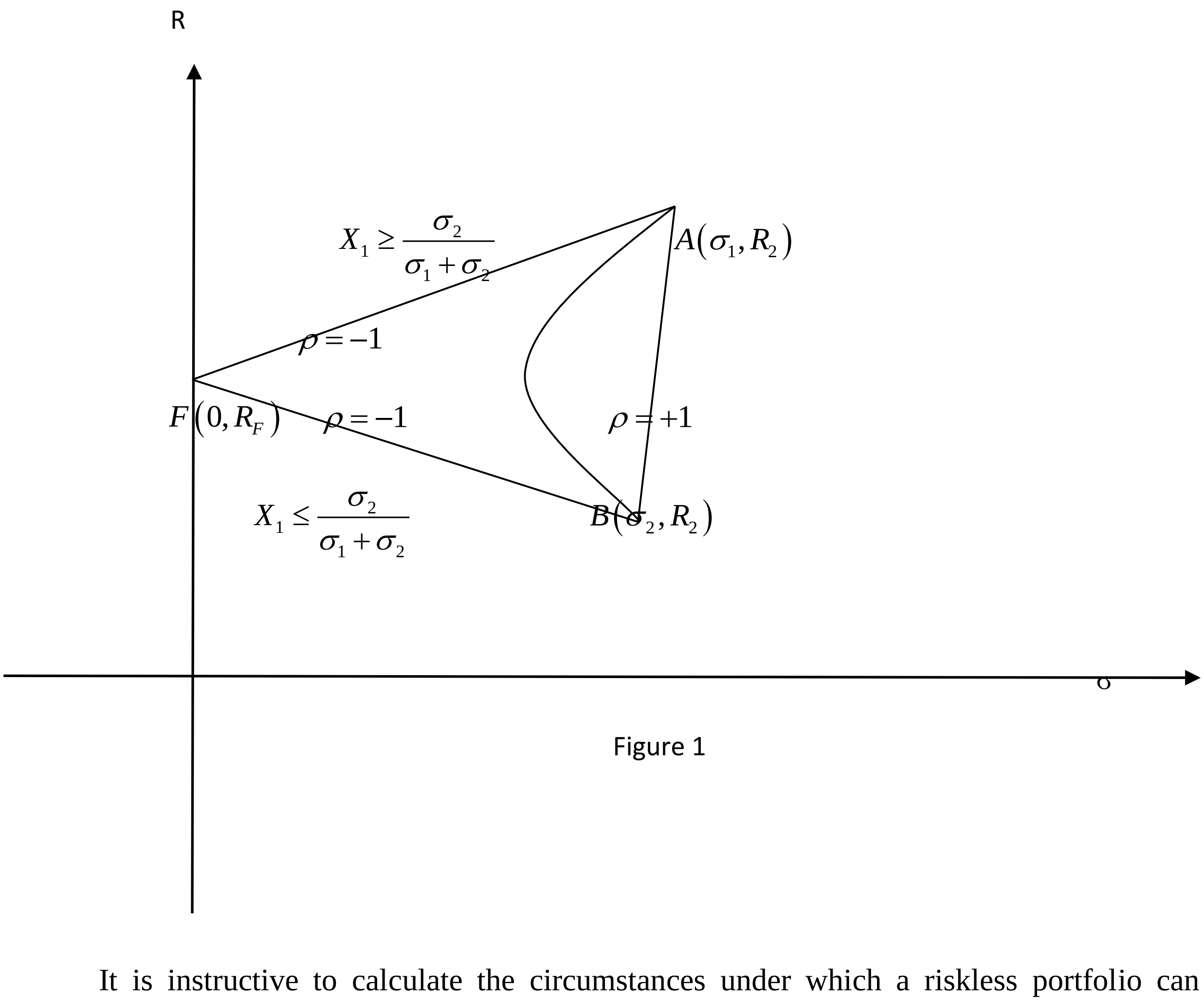

Figure 1

It is instructive to calculate the circumstances under which a riskless portfolio can be constructed from two risky assets. For the purpose, the PPC must intersect the ordinate axis at real points. In other words, the intersection of eq. (6) with $x=0$ i.e.

$$y^2\frac{\left(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2}-2y\frac{\left[R_2\sigma_1^2+R_1\sigma_2^2-\left(R_1+R_2\right)\rho\sigma_1\sigma_2\right]}{\left(R_1-R_2\right)^2}$$
$$+\frac{\left(R_2^2\sigma_1^2+R_1^2\sigma_2^2-2R_1R_2\rho\sigma_1\sigma_2\right)}{\left(R_1-R_2\right)^2}=0 \qquad (14)$$

must have real roots, the condition for which, on simplification, is found to be

$$\sigma_1^2\sigma_2^2\left(R_1-R_2\right)^2\left(\rho^2-1\right)\geq 0 \qquad (15)$$

yielding $\rho=\pm 1$ so that a risk free asset can be constructed out of two risky assets only if they are perfectly (anti) correlated. The case of perfectly correlated assets can yield a risk free asset only in the circumstances when short sales are permitted. This is seen from eq. (10). The ordinate intercept in that case is given by $R_F=\frac{\left(R_2\sigma_1-R_1\sigma_2\right)}{\left(\sigma_1-\sigma_2\right)}$ so that $R_F<\min\left(R_1,R_2\right)$ or $R_F>\max\left(R_1,R_2\right)$. For $R_P=X_1R_1+\left(1-X_1\right)R_2=R_F=\frac{\left(R_2\sigma_1-R_1\sigma_2\right)}{\left(\sigma_1-\sigma_2\right)}$, we obtain $X_1=\frac{\sigma_2}{\sigma_2-\sigma_1}<0$ (assuming $\sigma_1>\sigma_2$) implying short sales of security A since $X_1<0\Rightarrow X_2>1$.

<u>The PPC with Short Sales Permitted</u>

In the event when short sales are permitted, the components of the composition vector become unbounded $-\infty<X_i<\infty$ with the only constraint $X_1+X_2=1$. Hence, we can create portfolios with unbounded positive as well as negative returns (hypothetically) by short selling one or the other asset and investing the proceeds on the second asset. In such a case, the PPC gets extended beyond A,B along the same hyperbola i.e. the PPC consists of the entire section of the hyperbola that lies in the right hand side half plane bounded by the Y axis.

<u>The PPC with one of Securities being Riskfree</u>

Let the asset A, renamed F be a riskfree asset so that $\sigma_1=\rho=0$, $R_1=R_F$ so that eq. (6) for the PPC becomes

$$y=\pm\frac{R_2-R_F}{\sigma_2}x+R_F \qquad (16)$$

which is a pair of st lines that intersect each other and the Y axis at the point $F(0, R_F)$. Since x, being standard deviation must necessarily be positive, we have the following situation:

(a) If $R_2 - R_F > 0$, then the positive sign holds in eq. (16) in the region where $y - R_F > 0$ which corresponds to $X_F < 1$ i.e. no short sales of the risky security B. The PPC is the line segment FB terminating at the point $B(\sigma_2, R_2)$. The negative sign shall hold in the region where $y - R_F < 0$ corresponding to $X_F > 1$ that represents short sales of B and investment of the proceeds in the riskfree asset. With the possibility of unlimited short selling of B and investment of proceeds in F, the PPC in this case is the ray originating from F and extending to infinity with a slope that is the mirror image of FB. If $R_2 - R_F < 0$, the converse will hold i.e. the negative sign holds in eq. (16) in the region where $y - R_F > 0$ and vice versa.

(b) Let short sales of the riskfree asset i.e. riskfree borrowing be permitted, so that $X_F < 0$ becomes admissible. With the potential possibility of unlimited riskless borrowing and investing in the risky asset, the PPC, in this case does not terminate at the point $B(\sigma_2, R_2)$ but extends beyond B indefinitely along the straight line FB

<u>The PPC with two risky securities and a riskfree security</u>

Let $A(\sigma_1, R_1)$ & $B(\sigma_2, R_2)$ be two risky securities and $F(0, R_F)$ be a riskfree security.

(a) In the case when short sales is not permitted in either of the two risky securities and riskless borrowing is also not allowed, the PPC takes the form of a surface bounded by the straight line segments AF, BF and the arc of the hyperbola ACB. The line segment AF will represent combinations of A and F with B being absent and BF will represent combinations of B and F with A being absent. The arc of the hyperbola ACB will represent combinations of A and B exclusively. Any line segment CF will be a combination of all the three securities A, B & F where the relative proportion of A and B shall be determined by the location of C on ACB and that of F on the position of the portfolio point on CF.

It is pertinent to mention here that both straight line segments AF, BF shall intersect the closed arc ACB at no points other than A and B. This follows from (i) the point A must lie on the arc ACB since this arc represents portfolios of A and B that includes the portfolio of A alone and (ii) let, if possible, AF intersect ACB at another point D. Now, all points on the line segment AF must necessarily consist of only A and F. However, the security represented by the point D, that is common to ACB and AF shall consist of all the three securities, which is a contradiction.

(b) When short sales is permitted in A and B and riskless borrowing is not allowed, the PPC is determined as follows. We construct tangents from the point F to the arc of the hyperbola ACB, extended beyond A and B, if required. Let these tangents meet the extended arc of the hyperbola ACB at the points P and Q. The PPC, then consists of (i) the region PFQ being bounded by the straight line segments PF, QF and the arc PCQ (ii) the points on the arc of the hyperbola beyond CP, CQ extended indefinitely. The region PFA will represent combinations of the riskfree asset (long) with the asset A (long) and B (short) and similarly QFB will include A (short), B (long) together with the riskfree asset (long). Points within the region AFB will consist of combinations that are long in all the three securities. Points on CP, CQ extended beyond A, B respectively shall represent combinations of only A (long), B (short) and vice versa.

(c) When short sales is permitted in A and B and riskless borrowing is also allowed, the PPC is determined as in (b) above by constructing tangents from the point F to the arc of the hyperbola ACB, extended beyond A and B, if required intersecting ACB at the points P and Q. The PPC, then consists of the entire region of the positive X half plane that lies within the straight lines PF, QF extended indefinitely. In addition to the combinations explained in (b) above, points to the right of the arc PCQ shall contain riskless borrowing in addition to A and B while points lying in the region between the arc CP (extended) and FA (extended) beyond A represent combinations of riskless borrowing together with A (long) and B (short).

The coordinates of P and Q can be obtained in any of the following two ways:

(i) Let $y = mx + R_F$ be tangent to the hyperbola (14) so that it intersects the hyperbola at two coincident points, the condition for which is that the quadratic equation $x^2 - b\left(mx + R_F\right)^2 + 2f\left(mx + R_F\right) - c = 0$ must have equal roots which gives $m = \pm\sqrt{\dfrac{bR_F^2 - 2fR_F + c}{bc - f^2}}$ whence the equation of the two tangents is

$$y = \pm x\sqrt{\frac{bR_F^2 - 2fR_F + c}{bc - f^2}} + R_F \tag{17}$$

and the coordinates of P and Q are respectively given by

$$\left(\frac{m\left(bR_F - f\right)}{1 - bm^2}, \pm\frac{m^2\left(bR_F - f\right)}{1 - bm^2} + R_F\right) \tag{18}$$

(ii) The second method makes use of the fact that the tangents PF, QF maximize $\tan\theta = \dfrac{R_P - R_F}{\sigma_P}$. Making use of eqs. (4) & (5), we obtain

$$\tan\theta = \frac{X_1\left(R_1 - R_F\right) + X_2\left(R_2 - R_F\right)}{\left[X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\rho\sigma_1\sigma_2\right]^{1/2}} \tag{19}$$

Taking partial derivatives, with respect to X_1, X_2 and equating them to zero, writing $\frac{R_P - R_F}{\sigma_P^2} = \lambda$, $Z_k = \lambda X_k$, $Z_1 + Z_2 = \lambda$, we obtain the following eqs. for the composition vector:

$$R_1 - R_F = Z_1\sigma_1^2 + Z_2\rho\sigma_1\sigma_2 \tag{20}$$
$$R_2 - R_F = Z_2\sigma_1^2 + Z_1\rho\sigma_1\sigma_2 \tag{21}$$

which can be solved to obtain the composition vector $\mathbf{X}$ whence we can obtain the coordinates of P and Q.

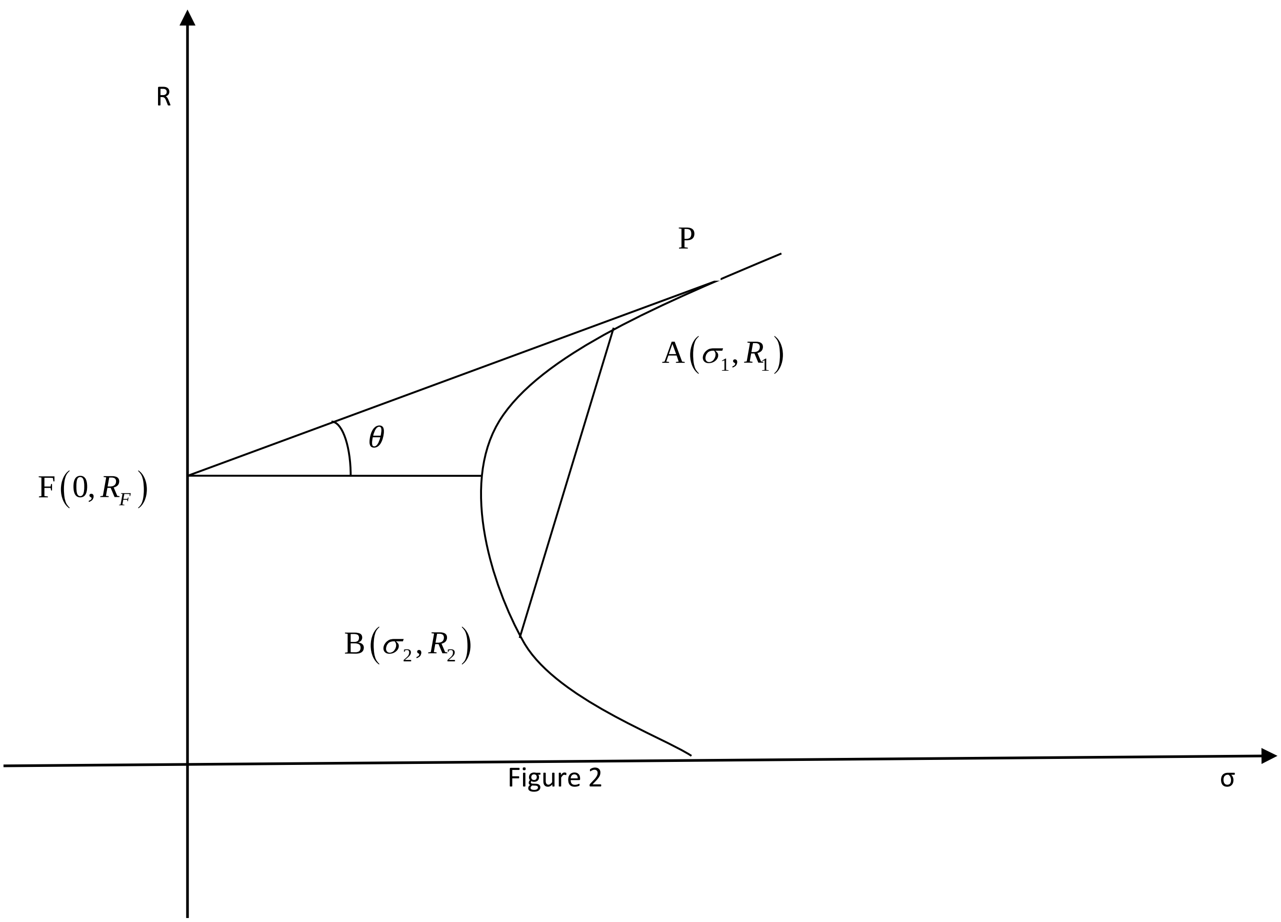

Figure 2

Concept of "Efficient Frontier"

To introduce the concept, we consider, first, the case of "no" short sales. Let $x = k$ be any line || Y-axis. Its intercepts with the PPC of eq. (6) are obtained by solving

$$k^2 - y^2 \frac{\left(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right)}{\left(R_1 - R_2\right)^2} + 2y \frac{\left[R_2\sigma_1^2 + R_1\sigma_2^2 - \left(R_1 + R_2\right)\rho\sigma_1\sigma_2\right]}{\left(R_1 - R_2\right)^2}$$

$$-\frac{\left(R_2^2\sigma_1^2 + R_1^2\sigma_2^2 - 2R_1R_2\rho\sigma_1\sigma_2\right)}{\left(R_1 - R_2\right)^2} = 0 \qquad (22)$$

This is a quadratic in y. For equal roots, we must have,

$$k^2 = \frac{\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = \sigma_M^2 \qquad (23)$$

showing that there is only one point such that the straight line || Y-axis is tangent to the PPC. Incidentally, this point coincides with the minimum variance point and the point of inflexion. Any other line || Y-axis shall intersect the PPC at two distinct points (real or imaginary). We are concerned here only with real points. Then, the segment of the PPC that lies between the point of minimum variance M and A (assuming $R_1 > R_2$) supersedes over the segment of the PPC lying between M and B in the sense that corresponding to every point on MB there exists a point on MA that provides a higher return for the same level of risk. Thus, the portion of the arc MA dominates over the portion MB and, hence, is called the "efficient frontier". The "efficient frontier" corresponding to various scenarios discussed above is tabulated here:

Scenario	**Efficient Frontier**
Two risky assets, no short sales, no riskfree asset	The arc of the hyperbola lying between the minimum variance point M and A (assuming $R_1 > R_2$)
Short sales allowed, no risky asset	The arc of the hyperbola from the minimum variance point M and extending through A (assuming $R_1 > R_2$) indefinitely
One risky asset with riskless lending	The straight line joining the riskfree asset and the risky asset in risk-return space
One risky asset and riskless lending & borrowing	The straight line extending from the riskfree asset through the risky asset to infinity in risk-return space
Two risky assets, no short sales, riskfree lending	The straight line segment joining the riskfree asset to the risky asset with higher return in risk-return space.
Two risky assets, short sales allowed, riskless lending	The straight line segment from the riskfree asset and tangent, with positive slope, to the arc of the hyperbola upto the point of contact. From that point on, the arc of the hyperbola that represents combinations of the two risky assets only.
Two risky assets, short sales allowed, riskless lending & borrowing allowed	The straight line segment from the riskfree asset and tangent, with positive slope, to the arc of the hyperbola extended indefinitely.

The Case of Three Risky Securities

In the case of two risky securities, the problem of tracing out the PPC is relatively simple because of its immediate compatibility with the two dimensional framework. However, an analysis of the three securities PPC elucidates some intriguing features of the portfolio optimization problem. We shall illustrate these features by means of an example to avoid getting lost in a plethora of calculations.

For the purpose, we consider three risky securities $A(6,14)$, $B(3,8)$ and $C(15,20)$ with $\rho_{AB} = 0.50$, $\rho_{BC} = 0.40$ and $\rho_{CA} = 0.20$ with the composition vector $\mathbf{X} = \{X_1, X_2, X_3\} \equiv \{1 - X_2 - X_3, X_2, X_3\}$. The equation of the PPC is obtained e.g. in terms of $x \equiv \sigma_P$, $y \equiv R_P$ and $z \equiv X_3$ by eliminating X_2 between the equations for expected return and standard deviation given by eqs. (2) & (3) and we obtain

$$x^2 - \frac{3}{4}y^2 - 306z^2 + 12y - 162z + 18yz - 57 = 0 \tag{24}$$

It is easily seen that the projection of the above curve on each of the three planes is a hyperbola. However, we need to focus on the XY plane. We can write eq. (24) as

$$\frac{x^2}{198z^2 + 18z + 9} - \frac{\left(\sqrt{\frac{3}{4}}y - \frac{18z + 12}{\sqrt{3}}\right)^2}{198z^2 + 18z + 9} = 1 \tag{25}$$

The above characteristics reveal that the projection of the PPC on the XY plane shall consist of a family of hyperbole $\{\mathrm{H}_z\}$ with each hyperbola corresponding to a value of $z \equiv X_3$. The centre of the hyperbola moves up along the Y axis as more of the security $C(15,20)$ is inducted into the portfolio and the point of inflexion also moves away from the abscissa as well as the ordinate axes showing that the minimum variance portfolio increases both in terms of the expected return and variance. Hence, the portfolio optimization problem, in essence, boils down to (i) identifying that hyperbola out of the family (of hyperbole $\{\mathrm{H}_z\}$) which is such that the value of $\tan\theta = \frac{R_P - R_F}{\sigma_P}$ i.e. slope of the tangent drawn from the riskfree asset to the hyperbola is maximum. Let this hyperbola be H_α; (ii) once the hyperbola is identified, to obtain the coordinates of the point of contact of the tangent (that has the maximum slope) with the hyperbola H_α (to which it is tangent). The efficient frontier then simply becomes the straight line joining the riskfree asset with the point of contact.

The procedure is purely an extension of the one set out in Section 7(c)(ii). Since a generalization to the N securities is absolutely straight forward, we set out the procedure for the latter, in view of its practical importance. Setting $\tan\theta = \frac{R_P - R_F}{\sigma_P}$, making use of eqs. (4) & (5), we obtain

$$\tan\theta = \frac{\sum_{i=1}^{N} X_i \left(R_i - R_F\right)}{\left[\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} X_i X_j \sigma_{ij}\right]^{1/2}} \tag{26}$$

Taking partial derivatives, with respect to each X_i and equating them to zero, writing $\frac{R_P - R_F}{\sigma_P^2} = \lambda$, $Z_k = \lambda X_k$, $\sum_{i=1}^{N} Z_i = \lambda$, we obtain the following eqs. for the composition vector:

$$R_i - R_F = Z_i \sigma_i^2 + \sum_{j=1,\, j \neq i}^{N} Z_j \sigma_{ij}, i = 1, 2, 3, \ldots, N \tag{27}$$

Thus, we get a set of N equations for an equal number of unknowns, being the components of the composition vector $\mathbf{X} = \left\{X_i, i = 1, 2, 3, \ldots, N\right\}$ which would, in the normal course, have a unique solution corresponding to the point of contact of the tangent to the hyperbola H_α identified as above. Knowing the composition vector, it is rudimentary to calculate the corresponding coordinates in risk-return space. The point so obtained would be the point of contact of the tangent of greatest slope with the hyperbola H_α. The efficient frontier is then, the straight line joining the riskfree asset with this point, extended to infinity, if riskless borrowing is permitted.