**Financial Derivatives and Risk Management**
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**Lecture 11: Futures Hedging: No of Contracts**

## Issue of lot size and optimum number of contracts

In fact, there are two issues that need to be considered in this context:

(i) The number of units in which an appropriate position should be established in the futures market to design the optimal hedge for a given exposure; and
(ii) Having determined the number of units that need to be shorted/longed in the futures market, the number of contracts that need to be traded for achievement of said optimal position.

In fact, the issue (ii) arise because futures are standardized contracts. They are standardized in terms of the contract size i.e. the number of units of the underlying that are covered by one contract. The lot size is prescribed in the terms of the futures contract by the exchanges when the contracts are released for trading. Lot sizes specified in the contract are not changeable during the life of the futures contracts. Thus, it becomes necessary to account for the integrality of the contracts when designing the hedge. Thus, given an optimal position in terms of the number of units to be positioned in the futures market, we may not be able to match that requirement precisely as trades in the market are in terms of lot sizes. Hence, we still need to determine how many contracts are to be traded for the optimal hedge. There would be a certain element of rounding off involved. This also contributes to the imperfection in futures hedging.

The issue at (i) is one which relates to matching of the hedge with the exposure i.e. what should be the number of units that should be taken by the hedger in the futures market to design the optimal hedge in context of his exposure. There are different approaches to this problem.

(i) The case of forward contracts is somewhat simplistic. The underlying is usually the same as the exposure. The number of units hedged by the forward contract coincides with exposure. The hedge maturity also coincides with the forward maturity. As a result, there is no basis risk and the hedging is perfect. The price at which hedged asset will be acquired/sold is known at hedge inception with certainty.

(ii) Let us, now look at a futures hedge. Let $Q_s$ be the exposure (in number of units), $S_0$ be the spot price per unit at hedge inception (t=0) and $S_N$ the spot price at hedge maturity (t=N). Let $F_0$ & $F_N$ be the corresponding futures prices. Let $Q_f$ be the quantity in which a futures position is created for the purpose of the hedge. Let $Q_f/Q_s=h$ (hedge ratio) so that $Q_f=hQ_s$. Then we have:

Value of the hedged portfolio at hedge inception t=0:

$V_{h,0}=Q_sS_0+Q_fF_0= Q_s[S_0+hF_0]$

Value of the hedged position at lifting of the hedged t=N:

$V_{h,N}= Q_sS_N+Q_fF_N = Q_s[S_N+hF_N]$

Change in value of the position t=0 to t=N: $\Delta V= Q_s(S_N-S_0)+Q_f(F_N-F_0)$ whence

$\Delta V= Q_s(S_N-S_0)+hQ_s(F_N-F_0)= Q_s[(S_N+hF_N)-(S_0+hF_0)]$

Now, if the hedge maturity coincides with futures maturity so that $S_N=F_N$ by convergence

$\Delta V= Q_s[(S_N+hF_N)-(S_0+hF_0)]= Q_s[S_N (1+h)-(S_0+hF_0)]$

If the hedge ratio =-1, then the above expression simplifies to $\Delta V=-Q_s(S_0-F_0)=-V_{0,h}$

Hence, value of the portfolio at hedge maturity $=V_{0,h}+\Delta V=0$

Thus, under the conditions: (i) underlying same as exposed asset (ii) hedge maturity coinciding with futures maturity and (iii) hedge ratio =1, we get a perfect futures hedge. Recall that these are conditions similar to what we had for the forward hedge.

(iii) In case (ii), we have been able to design a perfect hedge, the value of the portfolio at hedge maturity (t=N) is known with certainty at hedge inception (t=0). However, the conditions involved are much too rigid to be of serious practical utility in the context of futures hedging. We, therefore, look at an alternative approach viz the minimum variance hedge. We make the conscious assumption that variance (standard deviation) is an appropriate measure of risk. On that premise we work out the hedge ratio corresponding to the hedge of a given exposure such that said hedge ratio minimizes the variance of the hedged portfolio over the life of the hedge. We use the notation as in (ii) above. We have:

$\Delta V = Q_s\left(\tilde{S}_N - S_0\right) + Q_f\left(\tilde{F}_N - F_0\right) = Q_s\Delta\tilde{S} + Q_f\Delta\tilde{F}$ where the tilde represents random variables. Now, the variance of the hedged portfolio over the hedge period is:

$$\sigma_{\Delta V}^2 = Q_s^2\sigma_{\Delta S}^2 + Q_f^2\sigma_{\Delta F}^2 + 2\rho Q_s Q_f \sigma_{\Delta S}\sigma_{\Delta F}$$

For minima, $\frac{d\sigma_{\Delta V}^2}{dQ_f} = 0 = 2Q_f\sigma_{\Delta F}^2 + 2\rho Q_s\sigma_{\Delta S}\sigma_{\Delta F}\ Q_f = -\rho\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}}Q_s = -\beta_{\Delta S,\Delta F}Q_s$

Writing $\frac{Q_f}{Q_s} = h(hedge\ \ ratio) = -\rho\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} = -\beta_{\Delta S,\Delta F}$. Also $\frac{d^2\sigma_{\Delta V}^2}{dQ_f^2} = 2\sigma_{\Delta F}^2 > 0$ confirming that the above value of $Q_f$ corresponds to minimum variance. Thus, in the special case of minimum variance hedge ratio, we have:

$\frac{Q_f}{Q_s} = h(hedge\ \ ratio) = -\rho\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} = -\beta_{\Delta S,\Delta F}$ i.e. ***the hedge ratio in this special case of minimum variance hedge is the regression coefficient of the regression of changes in spot prices over the changes in futures prices.***

In the above, $S_0$, $F_0$ being spot and futures prices at t=0 are known at hedge inception. However, $S_N$, $F_N$ being future spot and futures prices at the projected date of hedge maturity are not known and, indeed, are random variables. It follows that $\Delta S=S_N-S_0$ and $\Delta F=F_N-F_0$ are also random variables. They ill have respective means, variances and probability distributions as well.

Note that the hedge ratio embeds a negative sign in this sign convention. As such $Q_f$ will be negative of $Q_s$ if $\rho$ is positive. Thus, for positive correlation between spot and futures, the positions in the futures markets will be opposite to that of the cash asset.

Note also that our degree of freedom i.e. the controllable variable in the entire formalism is the futures position i.e. the amount of underlying that we position in the futures for the hedge inception. The hedge is instituted by taking a position in the futures market. The amount of that oosition is, obviously, at the discretion of the hedger. Hence, in this formalism the hedger would choose that amount of futures units which would project a minimum variance of the hedged portfolio over the hedge period. Thus, the optimization (minimization) is done with respect to the futures quantity. Note that $Q_s$, $\rho$, $\sigma_{\Delta S}$,$\sigma_{\Delta F}$ are all given quantities and inputs to the model whereas the optimal $Q_f$ is the output.

Now, $h=Q_f/Q_s$ is the universal definition of hedge ratio in the context of futures. In the particular case, when we adopt a minimum variance hedge, this hedge ratio takes a corresponding optimal value (minimum variance hedge ratio) and that special value is given $h(\min imum\ \operatorname{var} iance\ hedge\ \ ratio) = -\rho\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} = -\beta_{\Delta S,\Delta F}$. When we do the hedging with the specific objective of minimizing the variance of the hedged portfolio over the hedge ratio, we need to adopt this special hedge ratio $h(\min imum\ \operatorname{var} iance\ hedge\ \ ratio) = -\rho\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} = -\beta_{\Delta S,\Delta F}$ and work out our exposure in the futures market for the hedge accordingly.

The following points need special mention in context of the above analysis:

(i) ***The regression is with changes in prices and not prices themselves.***
(ii) ***The futures price changes are the independent variable and the spot price changes is the dependent variable.***

Thus, the hedge ratio is the regression coefficient of the regression of ***changes in spot prices*** with respect to ***changes in futures prices*** and not between prices themselves. ΔF is the independent variable and ΔS is the dependent variable. Hence, we are looking at a regression equation of the form $\Delta S = \beta_{\Delta S,\Delta F}\Delta F + \alpha$ .

We have also not accounted for the integrality of the number of contracts so far. The lot size in futures market is standardized i.e. the number of units of underlying that are covered by one futures contract is standardized. One can trade only in an integral number of futures contracts and not in integral number of units of underlying. One futures contract constitutes the trading lot, trades have to be made in terms of the lot size, also known as the contract multiple.

**Perfect minimum variance hedge**

We have, for a minimum variance hedge $\frac{Q_f}{Q_s} = h\left(hedge\ \ ratio\right) = -\rho\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} = -\beta_{\Delta S,\Delta F}$ so that: $\left(\sigma_{\Delta V}^2\right)_{\min} = Q_s^2\sigma_{\Delta S}^2 + \left(-\rho\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}}Q_s\right)^2\sigma_{\Delta F}^2 + 2\rho Q_s\left(-\rho\frac{\sigma_{\Delta S}}{\sigma_{\Delta F}}Q_s\right)\sigma_{\Delta S}\sigma_{\Delta F} = \left(1-\rho^2\right)Q_s^2\sigma_{\Delta S}^2$

But, $\sigma_{\Delta V}^2 \geq 0$ always so that $\left(\sigma_{\Delta V}^2\right)_{\min} = 0$ at best whence $\rho = \pm 1$ for a perfect hedge. Thus, the minimum variance that can possibly be achieved under any circumstances by hedging is zero and that can be achieved when there is perfect correlation or perfect anti-correlation between the changes in futures prices and the changes in spot prices.

It is interesting to note that if the variance of a random variable is zero, then the random variation does not vary at all i.e. the random variable is simply a constant. If follows, therefore, that if the $\rho = \pm 1$ hedge can be formulated whence $\left(\sigma_{\Delta V}^2\right)_{\min} = 0$ i.e. the variance of the hedged portfolio is zero, the implication is that ΔV does not fluctuate at all i.e. it remains constant.

In view of the above, it is interesting to compare this situation with the scenario that we encountered in (ii) above. In (ii) above, we achieved a perfect futures hedge subject to the conditions that (i) the underlying was same as exposure (ii) hedge ratio was -1 and (iii) maturity of hedge and futures coincided.

Here we have somewhat relaxed conditions for a perfect hedge viz that (i) the underlying be same as the exposure (ii) $\rho = \pm 1$. But here also we get a perfect hedge (zero variance) albeit through a different approach.

For a perfect futures hedge, we must have: ρ= +1 or -1. If ρ= +1, the hedging position in the futures market will be opposite to the exposure and if ρ= -1, the hedging position will be same as the exposure.

If ρ=+1, then hedge ratio $Q_f/Q_s = h = -\sigma_{\Delta S}/\sigma_{\Delta F}$. Hence, if ρ= +1, but if the variances of fluctuations of the hedged and hedging asset are not equal, we can still scale the quantity in the futures market to account for the difference in fluctuation variances between the exposed asset and the hedging asset. Thus, we can still design a perfect hedge. The same holds for ρ=-1.

To reiterate, by scaling the quantity of the futures hedge, we can still create a perfect hedge if |ρ|=1, notwithstanding that $\sigma_{\Delta S} \neq \sigma_{\Delta F}$ i.e. even if the variance of the changes in spot prices and the futures prices are different provided that they are perfectly correlated.

Let us try to understand the difference between the two approaches (ii) & (iii) insofar as they relate to the perfect hedge. In (ii), the premise on which the hedge was instituted was the convergence between the spot and futures prices at futures maturity. The futures maturity was ensured to align with the hedge period. Thus, at the end of hedge period, the futures and spot prices converged i.e. $S_N=F_N$. Now, the risk in the exposure arises due to the randomness of the spot price at hedge maturity i.e. $S_N$. But because $S_N$ coincides with $F_N$, a position consisting of long cash+short futures or vice versa would no longer be random at t=N i.e. at hedge maturity. Thus, this approach simply focused at countering spot price randomness by futures price randomness at a given point in time (hedge lifting, t=N) because at that point in time, the two prices converged so whatever value the spot price took, futures price also took the same value. This was done by creating opposite positions via the hedge.

However, in the current approach, we are setting $\sigma^2_{\Delta V} = 0$ i.e. the variance of the hedged portfolio as zero. This implies that the change in hedged portfolio's value over the hedged period ΔV is constant. This is being achieved by using a hedge with |ρ|=1. Perfect (anti) correlation implies that the two variables are always in phase with proportional amplitudes so that if we construct a portfolio with opposite positions with appropriate quantities to account for the difference in amplitudes, then the net result will be mutual annulment in all cases. Whatever value one variable takes, the other variable will take a value with the same ratio so that because of opposite positions and appropriate quantities the net result will be zero. Let us illustrate this with an example. Consider the following data:

| ΔF | ΔS |
|---|---|
| 2 | 3 |
| 3 | 4.50 |
| 4 | 6 |

| | |
|---|---|
| 5 | 7.50 |
| | |

Clearly the correlation between $\Delta F$ and $\Delta S$ is +1. Suppose we construct a portfolio consisting of 150 units cover in the futures market (short hedge) and 100 units of our cash asset (long). Suppose, now, that the change in the spot price over the hedge period turns out to be -1 unit, then the loss on cash asset is -100. The loss per unit in the futures market is -1/1.5 so that the gain on the futures position of 150 units is -150*(-1/1.5)=+100 whence the change in value of the hedge is zero. Thus, in this approach a statistical formulation is adopted for the perfect hedge. The vital caveat here is that the actual change in futures price may not be exactly as dictated by the statistical relationship of $\rho=1$. $\rho=1$ is a statistical relationship that is obtained from analysis of historical data. Nevertheless, future price evolution does contain a random element, both in the context of spot prices and futures prices. Thus, the issue whether the futures & spot prices will maintain the historical alignment figuratively represented by $\rho=1$, is not mandatory. They are, of course, "expected" to follow this relationship i.e. "on the average" with a large number of observations, the futures and spot prices would tend to align towards $\rho=1$, but not necessarily for each observation.

From the practical perspective, we need to understand that the approach (ii) is more rigid but is more efficacious in the sense that if we implement that strategy the perfect hedge is a definite outcome and we know the hedge maturity and futures maturity at t=0 so that (at least theoretically) we can achieve matching. However, the second approach may seem easier and more practicable but the problem is that it is a statistical approach. It premises on $|\rho|=1$. The correlation would have been obtained on the basis of historical data but that does not, of itself, guarantee that the future evolution of prices will exactly follow the $|\rho|=1$ mandate, the only thing we can say is that on the average $|\rho|=1$ will hold, not that every future pair of price changes $\Delta S,\Delta F$ will necessarily follow the same proportionality. If this were so, the randomness would have been completely eliminated. Hence, this approach may not invariably yield a perfect hedge. It is not mandatory. This is a statistical relationship not an absolute one.

For $\rho$ having some other value e.g. 0.80, we have $Q_f/Q_s = h = 0.80\sigma_s/\sigma_f$. In such a situation, we find that since only 80% of the fluctuations are inter se correlated, only 80% of the risk of the exposure ($0.80\sigma_s$) is covered by the hedge. Accordingly, the hedging quantity is calculated with reference to this 80%. If the correlation is 80% or 0.80, only 80% of the standard deviation of the changes in spot prices is being used for working out the hedge position in the futures market. In other words, we are only trying to hedge 80% of the fluctuations of the exposed asset because we know only 80 percent are correlated with the futures prices.

The remaining 20% being unrelated to the hedging instrument continues to subsist in the overall hedged position. This is the reason that in such cases, the hedge is not perfect. $Q_f$ is also calculated with reference to the 80% correlation because any further increase in $Q_f$ will not enhance optimality. In fact, it will overshoot optimality.

So, the best we can do is to have an optimal hedge with respect to this 80% value of the changes in prices of the exposed asset. The remaining 20% will continue to be random or not be annulled by the hedge. The remaining 20% of the price fluctuations will not be covered by the hedge and will continue to be unattended.

The 20% of the risk is not hedge-able with this particular futures because there is no correlation between the prices of the cash asset and this particular hedging instrument to that extent. If there is 80% correlation, then 80% of the variation of the exposure prices will be hedge-able and the remaining 20% will remain unhedged if one uses that particular hedging instrument.

There is another very important caveat here, partially alluded to above. When we talk about hedging; we are talking about the future. We are talking about annulling the "future" price fluctuations of the exposure by taking an appropriate position in the futures market. The entire hedging process is forward looking, forward oriented.

Therefore, the inputs that go into the determination of the hedge ratio should ideally be forward-looking inputs. In other words, $\rho$, $\sigma_{\Delta S}$, $\sigma_{\Delta F}$, the three inputs that contribute to the calculation of the hedge ratio should, ideally, be forward looking. However, this is practically not possible and these statistical parameters are invariably premised on past data. Therefore, the underlying presumption is the past events; the past history is going to replicate itself in future.

To what extent this is likely to hold or not is a question which the hedger must assess when he takes up this hedging exercise. The hedger must use the past data after due analysis as to whether it is going to be relevant for the future. For example, the environmental conditions may have changed since when the data making the data outdated. So, if there is a material change in the environment, then cognizance has to be taken of the change and therefore, necessary adjustments to historical data need to be incorporated to take account of the changes in environment. Past patterns may not be replicated in the future; as a result of this the data being used for hedging computations may not be representative of the future course of events.

**Appendix**

## Variance of a linear combination fo random variables

Let X, Y be two random variables on the real line with means $\mu_X$, $\mu_Y$, variances $\sigma_X^2$, $\sigma_Y^2$ and $\alpha$,$\beta$ be some real numbers. Let Z=$\alpha$X+$\beta$Y. We need to find the variance of Z. We have:

$$\sigma_Z^2 = E\left[Z - E(Z)\right]^2 = E\left[(\alpha X + \beta Y) - E(\alpha X + \beta Y)\right]^2$$
$$= E\left[(\alpha X + \beta Y) - E(\alpha X) - E(\beta Y)\right]^2 = E\left[(\alpha X + \beta Y) - \alpha E(X) - \beta E(Y)\right]^2$$
$$= E\left\{\left[\alpha X - \alpha E(X)\right] + \left[\beta Y - \beta E(Y)\right]\right\}^2 = E\left\{\alpha\left[X - E(X)\right] + \beta\left[Y - E(Y)\right]\right\}^2$$
$$= E\left\{\alpha^2\left[X - E(X)\right]^2 + \beta^2\left[Y - E(Y)\right]^2 + 2\alpha\beta\left[X - E(X)\right]\left[Y - E(Y)\right]\right\}$$
$$= \alpha^2 E\left[X - E(X)\right]^2 + \beta^2 E\left[Y - E(Y)\right]^2 + 2\alpha\beta E\left\{\left[X - E(X)\right]\left[Y - E(Y)\right]\right\}$$
$$= \alpha^2\sigma_X^2 + \beta^2\sigma_Y^2 + 2\alpha\beta\sigma_{XY}$$