**MCDM Techniques Using R**
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**Lecture - 17**
**Introduction of Fuzzy Sets**

Welcome to the course MCDM Techniques Using R. So in this particular lecture we are going to start our discussion on another development in MCDM techniques which was the introduction of fuzzy sets. So the kind of you know, the kind of responses that we have to take from decision makers just like we talked about in previous technique like TOPSIS, VIKOR, ELECTRE and AHP certain subjective purposes are taken there. However, many of these techniques require decision makers to specify these preferences in more exact fashion.

So however, there might be scenarios where you know, decision makers might find it, you know quite difficult to give their preferences exactly. So those scenarios how they can be tackled and quantified and you know, analyze is something that can be done using Fuzzy sets. So let us start our discussion on this particular topic. So fuzzy sets, it is actually based on fuzzy set theory.

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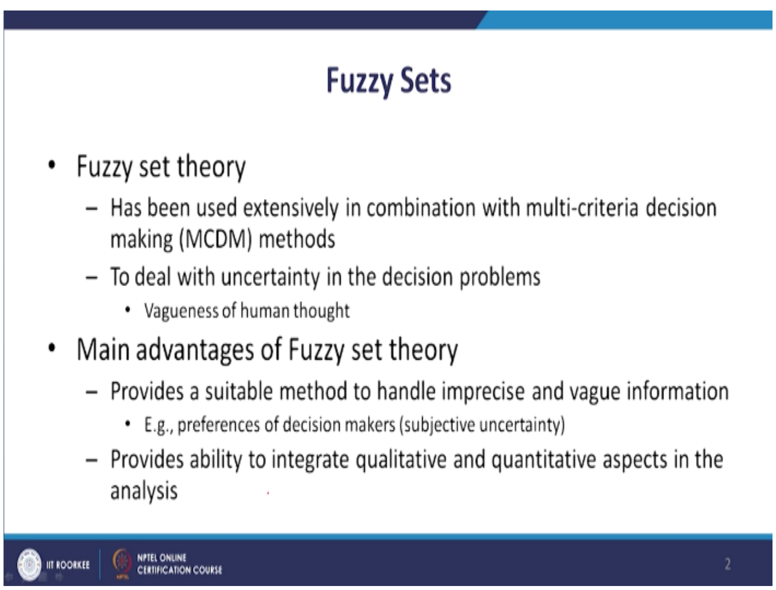


So this has been used extensively in combination with the MCDM methods. Main idea is to deal with the uncertainty in the decision problems. Vagueness of human thought, so just like I gave the example that decision makers might find it difficult to specify their subjective preferences at times. So to deal with that kind of, you know scenario fuzzy set theory can be really useful. So main advantages of fuzzy set theory are these two, provides a suitable method to handle imprecise and vague information where the exact comparison or exact you know, subjective references cannot be quantified exactly.

So in such a scenario fuzzy set theory can be useful. So as you can see in the slide preferences of decision maker is one example were subjective uncertainty is involved, so fuzzy set theory could be a suitable method to actually handle this impreciseness. Another one is that provides ability to integrate a qualitative and quantitative aspects in the analysis.

So fuzzy set theory it can integrate in that sense, the qualitative and quantitative both the aspects. Qualitative aspect using fuzzy theory can be quantified in and analyzed later on. Now, how the development of fuzzy sets have happened. So we are going to talk about few specific points.

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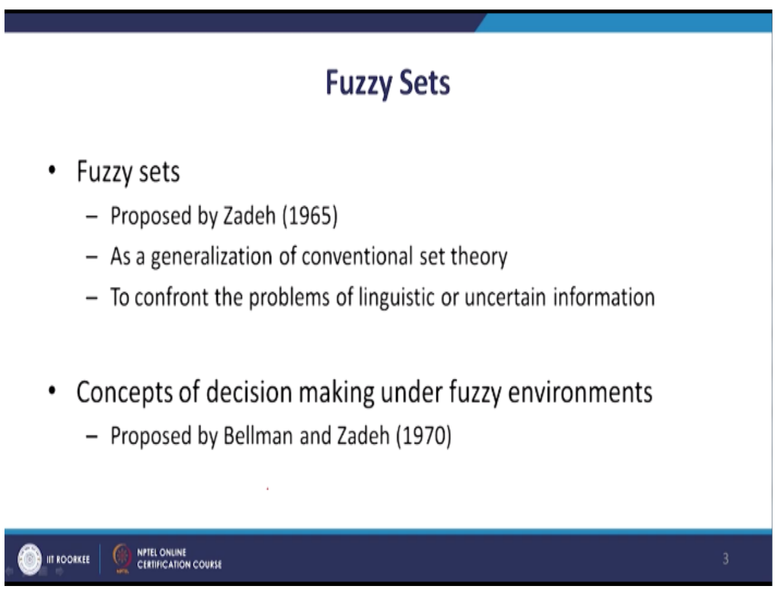


So fuzzy sets you know, they were proposed by Zadeh (1965) and this was more of as a generalization to conventional set theory that we are already familiar with. And the main idea is we have talked about to confirm the problems of linguistic or uncertain information. So either, if the subjective preferences are there and they cannot be specified clearly or exactly or you know, the way information is given in linguistic fashion and that is bringing the uncertainty.

So you know, the linguistic response and its quantification is not straightforward then that problem can be handled using fuzzy sets, further development in terms of concepts of decision making under fuzzy environments that was proposed by Bellman and Zadeh (1970). So based on these two developments, a number you know, researcher studies have been conducted which actually use fuzzy sets in combination with other techniques in MCDM. So to talk about more you know, related to fuzzy sets.

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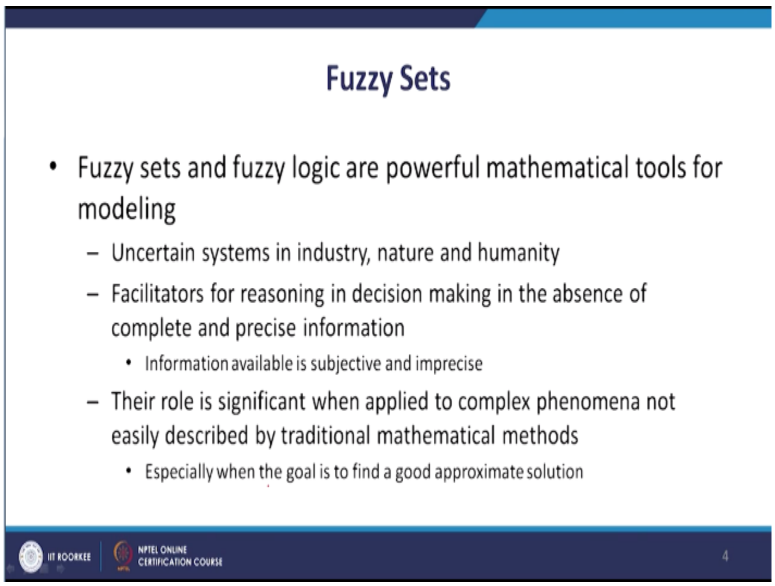


Fuzzy sets and fuzzy logic are powerful mathematical tools for modeling. So for example uncertain systems in industry, nature and the humanity, so there the fuzzy sets and fuzzy logic can be applied. They are also facilitated for reasoning in decision making in the absence of complete and the precise information, something that we have been talking about. Information available is subjective and imprecise then fuzzy sets and fuzzy logic are useful in modeling.

Their role is significant when applied to complex phenomena not easily described by traditional mathematical methods. So in that sense, so you know, fuzzy sets and fuzzy logic can really be useful. Especially when the goal is to find a good approximate solution which is typically the case in MCDM domain, when we talk about MCDM techniques typically we are looking for a good approximate solution. So fuzzy sets and fuzzy logic can be really useful. Now let us talk about some of the basic concepts of you know, fuzzy sets.

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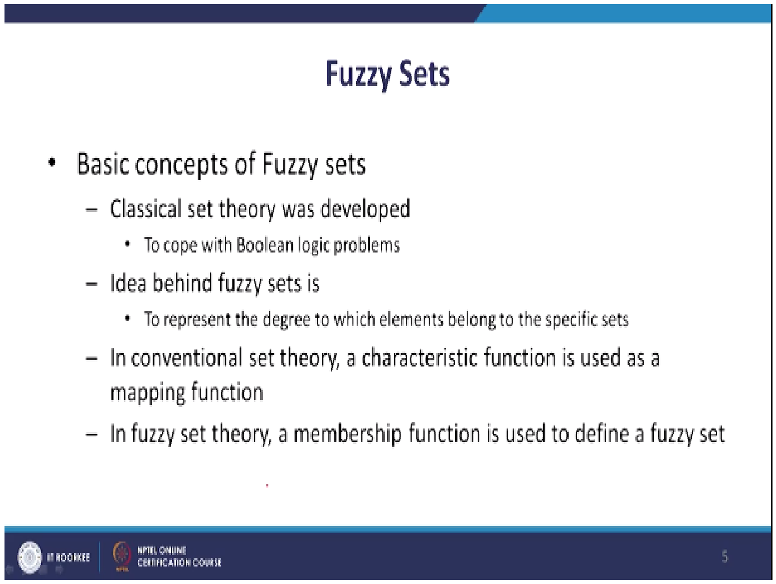


So you know, classical set theory as we know about, so that was actually developed to cope with Boolean logic problems. However, the idea behind fuzzy sets is to represent the degrees to which elements belong to the specific sets. So typically, in conventional theory, what we say is that a particular element it either belongs to completely belongs to the set or it does not belong. However, idea behind fuzzy set is you know, the element can partially belong to a particular set, and how that is to be quantified.

So it is actually indicated through a particular parameter which we are calling as referring as degree to which an element belongs to a specific set. Another difference with conventional set theory is that in conventional theory a characteristic function is used as mapping function. So whenever we talk about conventional set theory there is going to be a function which is going to map elements from set x to y. So that function is typically the; a characteristic function representing the main idea behind that particular function.

However, in fuzzy set theory we use a membership function so that membership function because here the idea is that, because we are alloying through fuzzy set, fuzzy set theory we are alloying membership to a particular set. So therefore, the main idea is actually the membership function, a function which can quantify this degree of belonging to a particular set. So that is why in fuzzy set theory a membership function is used to define a fuzzy set, so there is slight difference in that sense. More about fuzzy sets.

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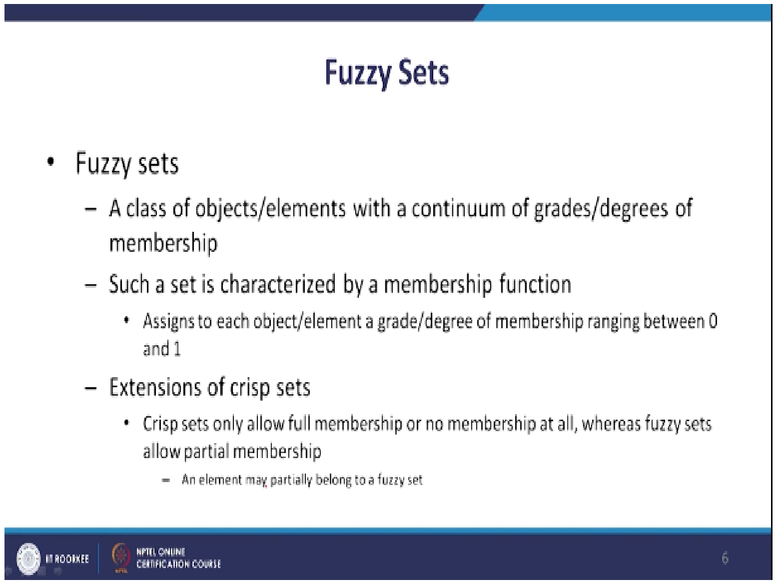


So we can say that a class of objects or element with a continuum of grades or degrees of membership. So there are going to be few elements which would completely belong to the particular set and there are going to be elements which are going to be partially belonging to the particular set. And then there are going to be elements which will not be belonging to set. So there is a going to be a continuum is going to be covered under fuzzy sets.

So that is the main idea. Now such a set is characterized by a membership function as we have talked about, so it assigns to each object or element a degree or grade of membership ranging between 0 and 1. So 0 meaning does not belong, 1 meaning you know complete membership and any value in between 0 and 1 indicates that, that partial membership is there.

Now a fuzzy sets they are also you know, in a way extensions of crisp sets, so crisp set is something that we have been using in this whatever we have discussed till now before this lecture in this course, we were actually using crisp set. Whenever we talked about a particular set in our underline mathematics in different techniques that we talked, we were always referring to crisp set.

So crisp set is the you know, the regular set that the way we understand them were a particular element is going to fully completely belong to the set or does not is not going to belong to the set. So that is what we refer by this set. So crisp sets only allow full membership or no membership at all. So that is the kind of sets that we have been using till now. Whereas, if we talk about fuzzy set they also allow partial membership as we discussed till now.

So therefore, an element may partially belong to a fuzzy set. Let us move forward. So how do we design a fuzzy set? So you can see the definition in the slide as well.

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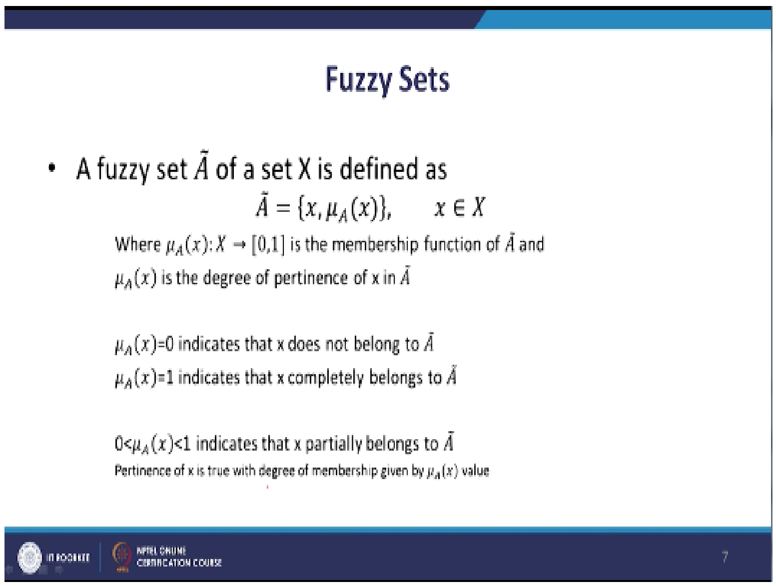


So a fuzzy set A tilde of a set capital X is defined as below, so where A tilde = x and then the membership value is given by Mu A function, and in parenthesis the element x which is going to belong to the set capital X. So this fuzzy set is actually you know, associated is going to be associated with a crisp set. Wherein a membership function is going to determine whether the full membership is there or you know no membership or partial membership is there.

So this is how we define, A tilde x, Mu Ax and for you know X belonging to capital X. So you can also see how the mapping happens. So for this particular function Mu Ax the elements from the set capital X, they are map to by this range 0 to 1, so this is the membership Mu Ax is referred as Mu A is referred as the membership function of A tilde and as we have talked about, so this is actually this Mu Ax is actually indicates the degree of pertinence of an element x in A tilde.

So whether the element X that is coming from the set capital X, what is the degree of pertinence in the sense what is the degree of membership of this particular element x in the fuzzy set A tilde. So if this membership value this you know, value from this function comes out to be 0 that is Mu Ax is 0 then it indicates that X does not belong to A tilde. If this Mu Ax value is 1 then it indicates that X completely belongs to A tilde, and if the value Mu X lies between 0 and 1, so that indicates that X partially belongs to A tilde.

The pertinence of x is true with degree of membership given by x. So the value that, so when for any given element x belonging to set capital X, so when we apply this function Mu A there and the value that we get so that value is actually indicating the degree of membership or you know pertinence, whether the pertinence is there.

So this is how we define a fuzzy set A tilde for a given set. So let us move forward. So when we talked about the; a linguistic variables you know when they are used and the uncertainty is there, impreciseness is there, so fuzzy sets can be really useful. So how all this is actually done in fuzzy sets, so we will talk about of that.

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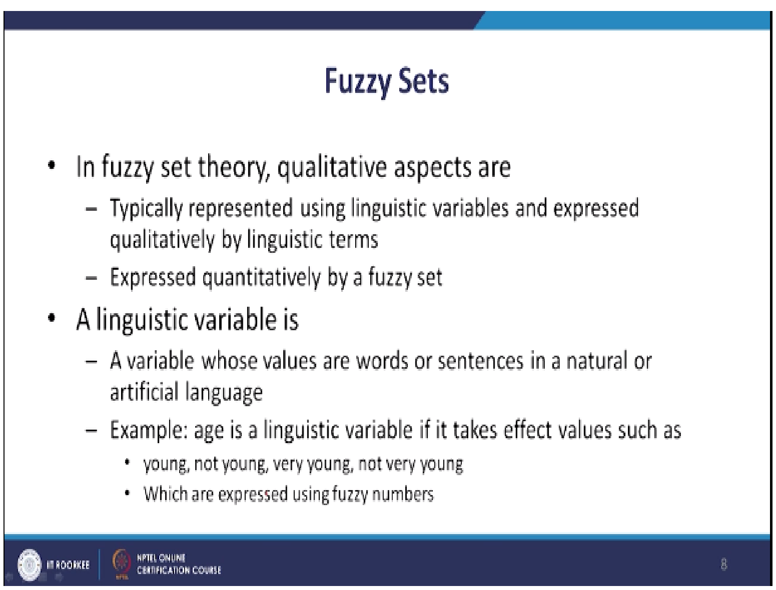


So in fuzzy set theory qualitative aspects which are typically represented using linguistic variables and expressed qualitative using linguistic term, so we will talk about some of the example, how this is done later in the lecture. So you know, these qualitative aspects they are represented using linguistic variables, qualitative expressed using linguistic terms and quantitatively expressed by a fuzzy set.

So the definition that we saw in the previous slide and how the fuzzy set is defined, so that is actually you know definition is actually used to express the qualitative aspect in a quantitative manner. So let us talk about what we mean by a linguistic variables, so linguistic variable as you can see in the slide is a variable whose values are words or sentences in a natural or artificial language. So the words and, words or sentences that we are going to use, so sentences are could be half a sentence kind of thing.

So these words are sentences or the values, so of course they are going to be structured because we are in the decision making domain, so we are going to take responses for decision makers so we have to take the responses in a more structured manner, so that is why you know the linguistic terms that are going to be used the words or half a sentences or sentences that we are going to use to elicit the responses from decision makers, so they are going to be more structured.

So therefore, you know the definition also refers to a natural or artificial language so that is there. So for example, age is a linguistic variable if it takes effect values such as young, not young, very young, not very young. So these are some of the terms which are in a way expressing indicating the age of an individual or it could be even for a firm, so young, not young, very young, not very young. So if you look at the terms that are being used here you would see the impreciseness there itself.

So the way linguistic terms are used, the way decision makers are going to respond to a particular question, there is going to be a certain level of impreciseness. That is why the use of fuzzy sets in this is important in MCDM domain. So these could be example, age, young, not young, very young, not very young and which are expressed using fuzzy numbers. Now these terms can be map to fuzzy numbers just like the definition of fuzzy set we saw in previous slide.

So let us move forward. So now that we know the definition of how the linguistic variable is defined, so what is the main idea, what is the main concept of linguistic variable?

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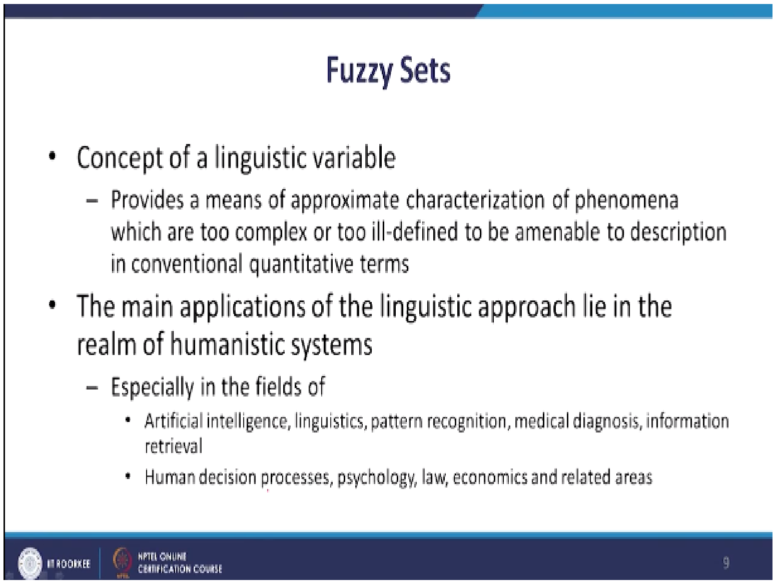


So it provides a mean of approximate characterization of phenomena which are too complex or too ill-defined to be amenable to description in conventional quantitative terms. So as we have talked about if there is impreciseness, if there is uncertainty something is too difficult for decision makers to specify clearly or directly then this concept of linguistic variable, this gives them an opportunity to and us as well as analyst an opportunity to approximately characterize a phenomena.

If the exact characterization is not possible and at least we should be able to go ahead with the approximate characterization of the phenomena, so fuzzy sets they you know, enable us to do this and the concept of linguistic variable also facilitates this. So we look at the applications of the linguistic approach, so many of the humanistic systems were human decision making is involved which covers a whole lot of space, so there, you know this can be applied.

So you can see the fields which are mentioned here in slide. The artificial intelligence, linguistic itself, pattern recognition, medical diagnosis, information retrieval so these are some of the, these are some of the domains which are more technology oriented where the linguistic approach can be applied. Then we can talk about human decision processes, psychology, law, economics and other related areas where this linguistic approach can actually be used.

Now another related concept you know with fuzzy sets is fuzzy numbers. So we have already defined what we mean by a fuzzy set. Now let us talk about the fuzzy numbers.

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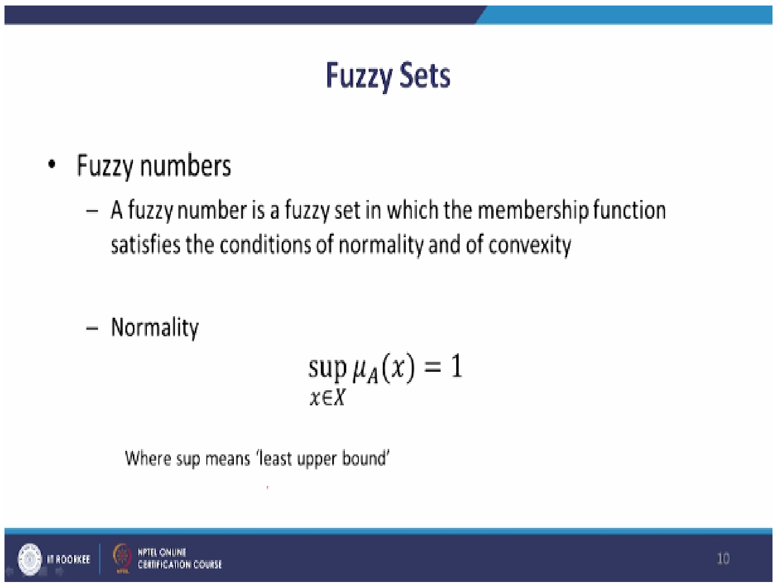


So fuzzy numbers something that we are actually going to use in our in combination with our MCDM techniques. So let us understand what mean by fuzzy numbers. So a fuzzy number is a fuzzy set in which the membership function satisfies the conditions of normality and of convexity. So fuzzy number is also a defined as a fuzzy set only, but with certain conditions, certain conditions are to be satisfied that is first one being that of normality and of convexity. So what do we mean by normality here?

So by normality we mean this, so Mu A is the membership function that we have already talked about. So you know, for any element x we apply this function Mu A, so the value of Mu Ax and we take S-U-P sup of this value given x belonging to capital X then this value should be 1, so whereby this function sup, S-U-P we actually mean is that least upper bound.

So what it essentially means is that there should be you know, one you know, let us say x dash element in the set where this value, membership value comes out to be you know 1, so this is what we are referring as lead upper bound. So this should be 1, then; so it is essentially also indicating towards that at least one element should have the complete membership kind of scenario. So this is what we are referring as normality. So if this is not the case, if the value is not 1 then the, this you know, fuzzy set is not normal and therefore it cannot be defined as a, cannot be defined as a fuzzy number. So this should be 1 this value should be 1.

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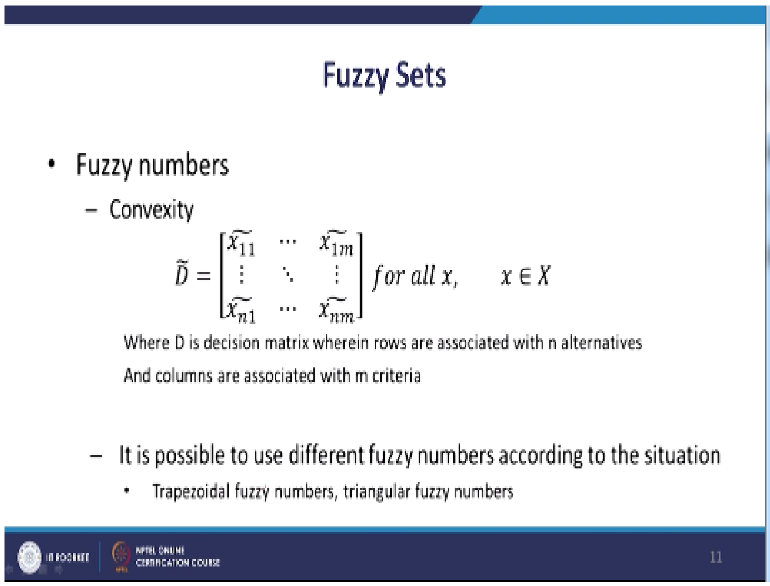


Then another condition is convexity. So if you look at this condition so we are defining it like this, so there is this D is a decision, capital D is a decision matrix where you know typical decision matrix that we have been using in other lectures as well other techniques as well. When rows are associated with the alternatives and columns are associated with the you know criteria. So n alternatives along the row side and m criteria along the column side, so this is decision matrix n cross m, so we are denoting it right now as capital D.

Now D tilde is going to be define a matrix like this where every element would be like this, like this x11 tilde x1m, tilde and this is xn1 tilde up to xnm tilde. So this is for all where X you know belongs to the set capital x. So this would be possible, so if this is possible all these you know, all these, x for all these x values x elements belonging to capital X, if we can drive these tilde versions of these elements and you know define our decision matrix then that you know satisfied the condition of convexity.

So you know, so another aspect related to fuzzy numbers is that there could be different way fuzzy numbers can be defined further according to situation. So there two popular forms that are available are Trapezoidal fuzzy numbers and triangular fuzzy numbers. So the triangular fuzzy numbers are the one which have been used more often. We will understand what we mean by triangular fuzzy numbers.

But before we go ahead two conditions, so given a fuzzy set two condition are normality and convexity they are two be satisfied so we can consider them fuzzy number and then there are various forms of fuzzy numbers. And they can be used depending on the situation or the depending on the decision problem and the overall context. So now we are going to talk about more about the triangular fuzzy number.

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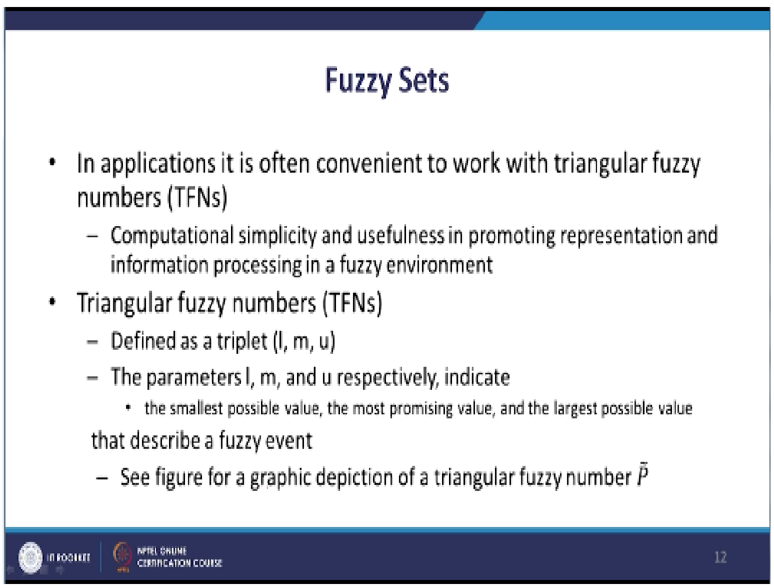


So in applications it is often convenient to work with triangular fuzzy numbers, so main reason being that computational simplicity. And usefulness in terms of promoting representation and processing in fuzzy environment, because the way we are going to define these fuzzy numbers it is going to put a cost on the processing, the computing that is going to be involved and the way we are going to represent the you know, overall decision problem and context and method procedure and solution and thereof.

So therefore, you know triangular fuzzy number mean you know, idea being that computationally simple and easy to represent. So that being the main idea. They have become more popular. So now let us talk about triangular fuzzy numbers. So in brief they are also referred as TFNs.

So they defined as a triplet, so where we have three alternates here I, m, u, so where these parameters 1, m, u or respectively they are defined as the smallest possible value that is indicated by L then the post promising value that is indicated by M and then the largest possible value that is indicated by U. So as we have talked about that when we talked about, you know fuzzy set in more general sense that the idea is to even allow partial membership of elements. So if you look at the definition of TFNs Triangular Fuzzy Number were they are defined, the parameters that are being use to define them l, m, u, and the smallest possible value, the most promising value, and the largest possible value.

So in a sense conveying this you know, notion of partial membership, right. So these parameters they are used to refine triangular fuzzy numbers. And then these you know, with the help of these triangular fuzzy numbers we can describe a fuzzy event. So any phenomena which is having a certain level of uncertainty and impreciseness and therefore can described using fuzzy numbers. So we are referring to that phenomenon as a fuzzy event. So let us look at graphic defection of triangular fuzzy number to understand these parameters with more clarity.

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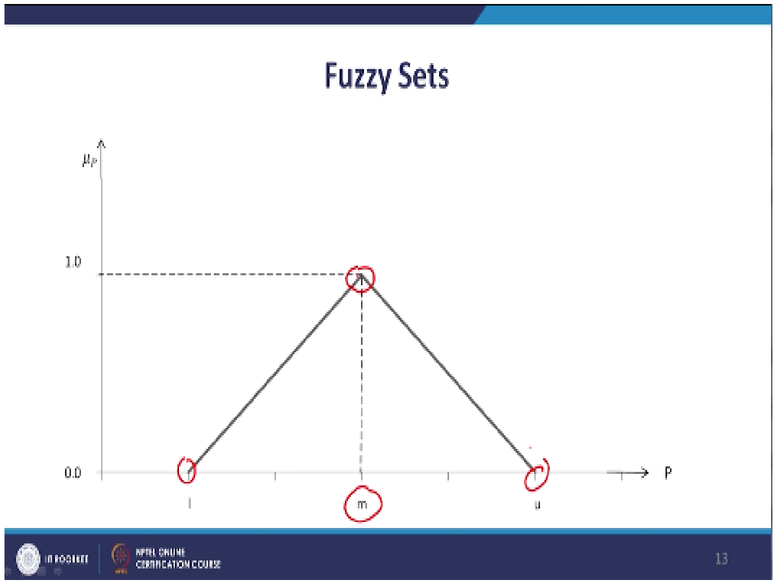


So this P tilde is a triangular fuzzy number. So in this graphic it is depicted. So there in the y-axis you can see the membership function for this, because these numbers are also set, so Mu P is the membership function values along the y-axis you can see as we have talked about it takes values between 0 to 1 so the same is depicted on y-axis and then we talked about that these numbers they will have you know, three values l, m, u and l is going to be you know, always less than m and m is going to be less than u.

So this is how it can be depicted in a triangular fashion. So you can see here. So along the y-axis we have this set p, so where these values this parameter values l, m, u so they coming from this set so therefore, they can be any wired value, right because this p is a set in more conventional form.

So this is how a triangular fuzzy number can actually be depicted graphically, where you can see here this m being the you know, most promising value, so here, so the you know, membership function value is also indicating that and then this is low then the l is actually the smallest possible value indicated here and then u is the largest possible value.

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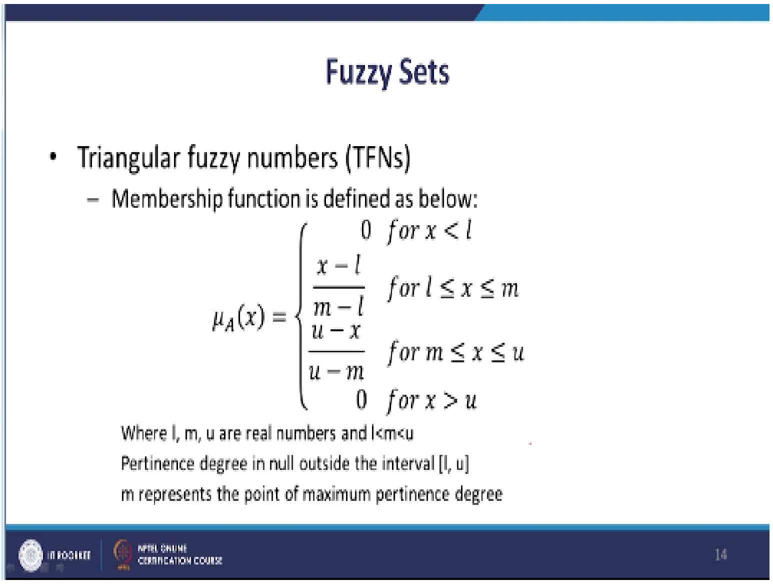


So let us move forward. So given our understanding of triangular fuzzy numbers how they are defined, how membership function can be defined further, so this Mu Ax can be defined as 0 for any value for any element x less than l. And then you know for the value x lying between l and m. so we go back to slide, so if the value is lying somewhere here then this is given by x-l this ratio x-l in the numerator and the denominator m-l.

So this will actually indicate the membership value. Then if the value is if the element x is lying between m and u so we go back to the previous graphic so the value is here lying in between these two parameters. Then the membership value is going to be this, ratio and the numerator u-x/ in the denominator u-m.

And if the x is greater than u then again the value is going to be 0. So as you can see l, m and u these three parameters they are real numbers and l is going to be less than m which is going to be less than u. So the pertinence degree is null outside this interval l and u. So as you can see in the definition itself, 0 for x less than l and 0 for x greater than u. So there is going to be no pertinence outside this interval. And m represents the point of maximum pertinence degree, right. So let us move forward.

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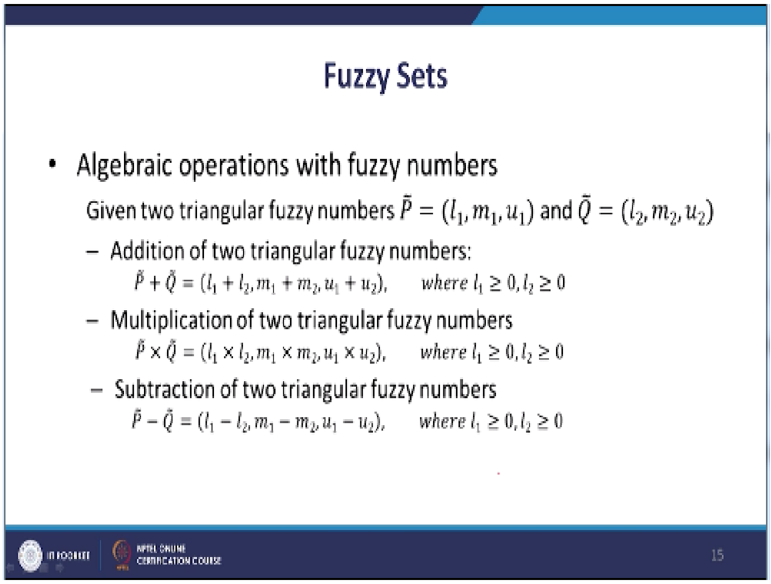


And now what we are going to discuss is the Algebraic operations with fuzzy numbers. So now that we are familiar with how fuzzy numbers are defined. And how you know, through the graphic we also understood how the; this triplets l, m, u these parameters, how they are defined, how they are going to be used. So knowing all these we can go ahead and talk about the algebraic operations that can be performed using this fuzzy numbers. But we will stop at this point and we will continue our discussion on fuzzy sets in the lecture. Thank you.