Decision Making Under Uncertainty
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Lecture – 07
Expected Value: Mean, Variance and Functions

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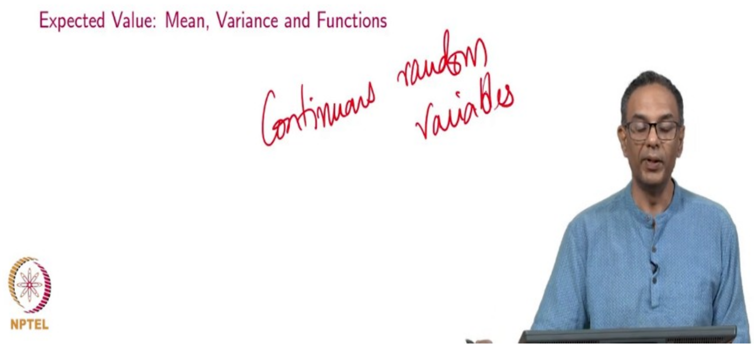

Now, we will talk about computing the Expected Value and variance of continuous random variable. So, this is essentially for continuous, not discrete. We have already seen the discrete random variable case. We will only look at continuous random variables here, alright.

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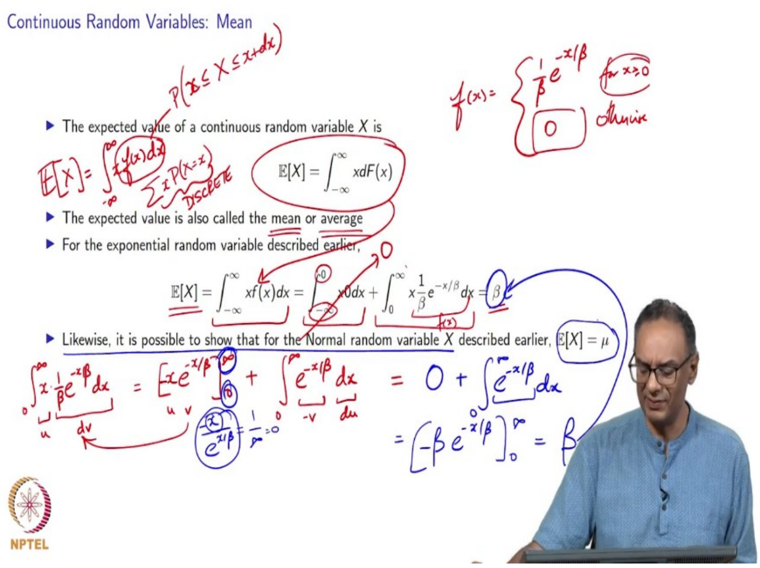

So, we are going to be talking about the mean first and then, we will spend the next slide talking about the variance. So, by definition, the expected value of the continuous random variable is given by this, sometimes we also write this as the $E(X)=\int_{-\infty}^{\infty} xf(x)dx$. Now, I do not want to keep jumping between the PDF and the CDF. So, I tend to write everything in terms of CDF, because the rest of the course is probably going to be mostly CDF oriented where $E(X)=\int_{-\infty}^{\infty} x\,dF(x)$, these two are basically equivalent.

So, that is how we define the expected value. Mathematically, this is very similar to what we had before for the discrete case, $E(X)=\sum_x x\,P(X=x)$. So, this was in the case of discrete. In the continuous case, it is the same situation and this guy, $f(x)dx$ is the same as the probability that the random variable X is between x and $x+dx$, if you will. So, if you sum over all values, it is the same flavor. So, in some sense, we are talking about the same thing; they are not 2 different things, but I just wanted to clarify that. So now, what we will do is we will use in this notation and compute it. So, the expected value again is called the mean or average and we will interchangeably use these three words; sometimes, call it expected value; sometimes, call it mean; sometimes, it's called the average. You have to be very careful; again, the probability that it takes exactly one value is 0. So, it's not like a value that we expect to see. So, let us do one example; I will do in great detail and will tell you a little bit about the other. So, we saw the exponential random variable earlier and we said the expected value of the exponential random variable is β. In fact, what we said was that we picked the exponential random variable with parameter β where β was the mean. We are essentially going to derive that.

So, based on the result that we had above, you directly get this step. It is just by definition, $\int_{-\infty}^{\infty} xf(x)\,dx$. But, if you remember, the little $f(x)$ for the exponential distribution looks like $\frac{1}{\beta}e^{\frac{-x}{\beta}}$ for $x \geq 0$ and it is equal to 0 when $x<0$. So, I have my integral split into 2 parts: the first part, I do $\int_{-\infty}^{0} x\,0\,dx$ and then, plus $\int_{0}^{\infty} xf(x)\,dx$. So, we want to show this result equals β. The first term is 0 because it is x times 0; so, that goes away. So, what we are left with is, and I have not done the calculation.

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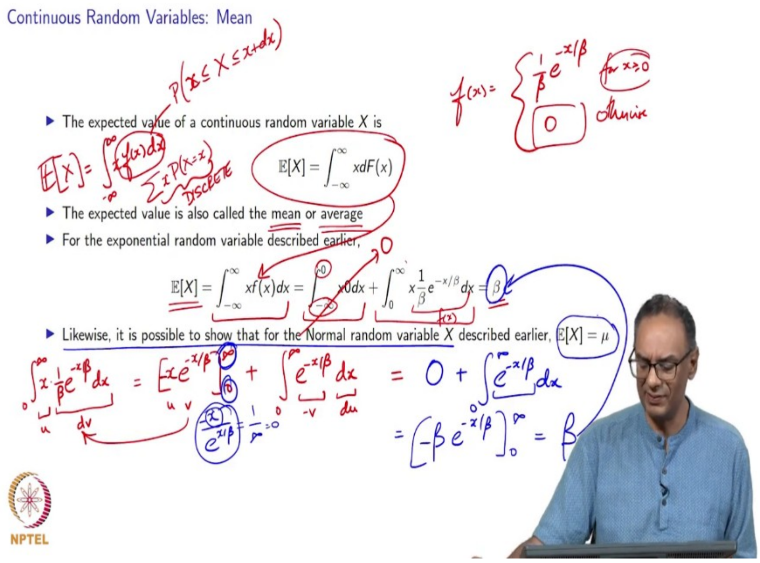

I am going to do that here, $\int_{0}^{\infty} x\frac{1}{\beta}e^{\frac{-x}{\beta}}\,dx$. So, we want to perform this integral. The moment we see something like this, we are tempted to use integration by parts, which is the right thing to do. So, this would be u times v. So, uv would be $[-xe^{\frac{-x}{\beta}}]_0^\infty$ right. If you take the derivative of the v, you get dv. So, $-v\,du$ would be $\int_0^\infty e^{\frac{-x}{\beta}}dx$. So, $\int_0^\infty x\frac{1}{\beta}e^{\frac{-x}{\beta}}dx=[-xe^{\frac{-x}{\beta}}]_0^\infty+\int_0^\infty e^{\frac{-x}{\beta}}dx$.

Now, an important thing that we want to talk about is how we calculate these limits here. Consider the first term, $[-xe^{\frac{-x}{\beta}}]_0^\infty$. Now, notice that when I plug in 0, I get $0e^0=1$; therefore, 0 times 1 is 0; that is the easy part. The infinite is a little bit tricky, alright. Typically, the way we write that down is to write this as $\frac{-x}{e^{\frac{x}{\beta}}}$ and then, realize that if I put ∞, their numerator and denominator both will become ∞. And then, I use L'Hôpital's rule once. Then, I would get $\frac{1}{\infty}=0$. Therefore, I would get the first term as 0, because when I plug in ∞, I get 0 and when I plug in for 0, I get 0; therefore, it is equal to 0. The another way of thinking about this is x climbed slowly to ∞ whereas $e^{\frac{-x}{\beta}}$ crashes down to 0 really quick for even tiny values of x like 4 and 5. So, the growth of x is much slower than the decay of $xe^{\frac{-x}{\beta}}$. So therefore, the product is 0. You can think of it in either ways.

Now, the other part is $+\int_0^\infty e^{\frac{-x}{\beta}}dx$ and we want to compute that. Well, if you look at this, this is a straightforward integral, which is equal to $[-\beta e^{\frac{-x}{\beta}}]_0^\infty$ because take a derivative of this, you get this quantity. So, this from 0 to ∞. When I plug in ∞, I get 0; when I plug in 0, I get 1. And, I get β. So, $\int_0^\infty x\frac{1}{\beta}e^{\frac{-x}{\beta}}dx=[-xe^{\frac{-x}{\beta}}]_0^\infty+\int_0^\infty e^{\frac{-x}{\beta}}dx=\beta$. So, this is how we derive this expression.

Similar fashion, but much harder, is to show that the normal random variable X if you compute it, you need to do integration by substitution, and nice little substitution will give me this result.

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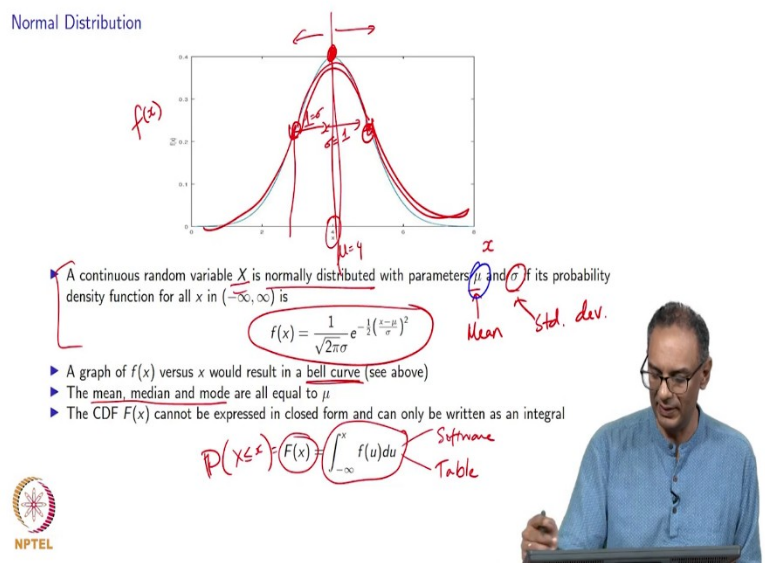

Now, we had already seen this result earlier we already said that the mean of the random variable is, mean, median and mode are all equal to μ; we said that earlier. So, we are going to use that exact same result right here; so, we get that result.

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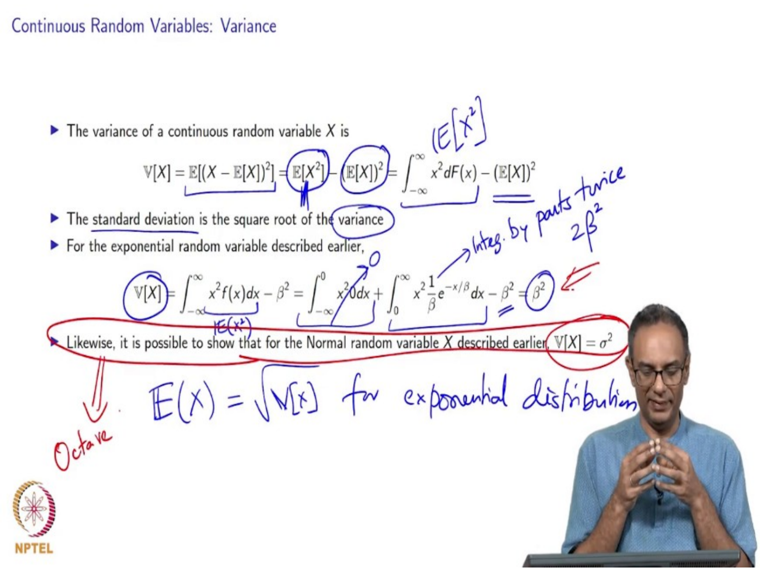

Now, we move on to the next item which is called variance. The variance of a continuous random variable much like the discrete, is equal to $E\left[\left(X-E[X]\right)^2\right]=E\left[X^2\right]-\left(E[X]\right)^2$. Again, we write that down and we will see this again in the next slide as $\int_{-\infty}^{\infty} x^2 dF(x) = E[X^2]$, it becomes $E[(X - E[X])^2] = \int_{-\infty}^{\infty} x^2 dF(x) - (E[X])^2$.

So, how do we go about doing that? Well, we will see a little example, but before that, I do want to say that the standard deviation as a quantity is just a square root of the variance. Now, this by definition, is always true for discrete or continuous, so, that does not change. Now, for the exponential case, I will not do the calculation like I did before; for the exponential case, I want to compute the variance. I first write this down.

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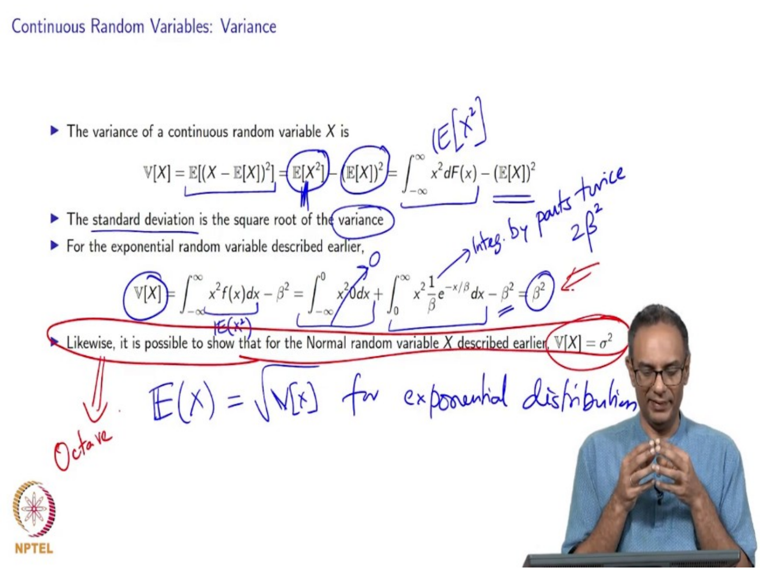

So, the variance of X, $V[X] = \int_{-\infty}^{\infty} x^2 f(x) dx - \beta^2$, where $\int_{-\infty}^{\infty} x^2 f(x) dx = E[X^2]$ and $\beta^2 = (E[X])^2$.

So, I go through the integral, $\int_{-\infty}^{\infty} x^2 f(x) dx - \beta^2 = \int_{-\infty}^{0} x^2 f(x) dx + \int_{0}^{\infty} x^2 f(x) dx - \beta^2$. So, the first term becomes 0 and then, for the second term, I need to do integration by parts 2 times. When I do integration by parts, I do integration by parts twice and I highly recommend that you do this if you have not done it before; perform the integration by parts 2 times and you should get the answer for the second term as $2\beta^2$. Once you get the $2\beta^2$, then subtract off a β^2giving you β^2.

So, notice that the expected value of X equals the square root of the variance of X for exponential distribution; this is a very special distribution whose mean and standard deviation are equal. However, for a random variable such as the normal distribution, that is not the case and we have seen this before. The normal distribution basically has a variance of σ^2, we saw that before. And, it should not be surprising; you can again carry over the integral, little trickier than the exponential. Of course, all these results are also available online; you could look at Khan academy; you could look at several sites online; there are many people that have derived it.

So, if you are not able to derive this guy or the previous expected value or the variance here, please go check one of the web sites; they will do a great job. Now, I do want to spend a little bit time talking about doing things on Octave and I am going to do a little demo to show you how different values of σ affect the random numbers that are generated. So now, what I am trying to say by this is, we want to understand what is this whole deal of variance. So, when a random variable has higher variability, the numbers also start to get more and more varying.

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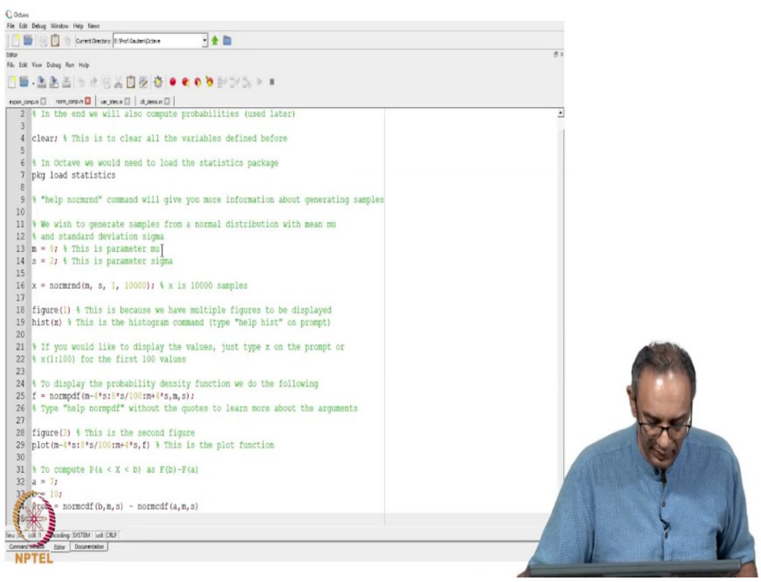

So, let’s look at that. I am going back to Octave.

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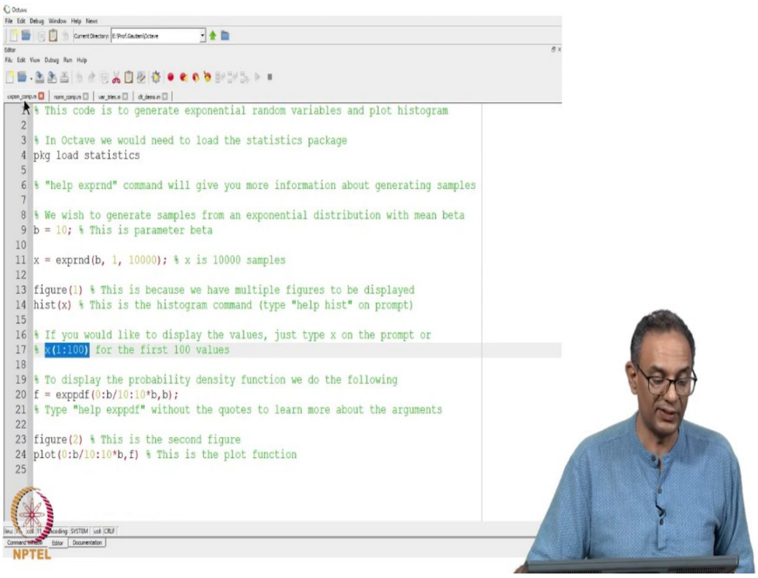

So, notice that my first file is called expon_comp.m; again, this is like MATLAB. So, that is where it has a dot m next one is the norm comp which we saw this.

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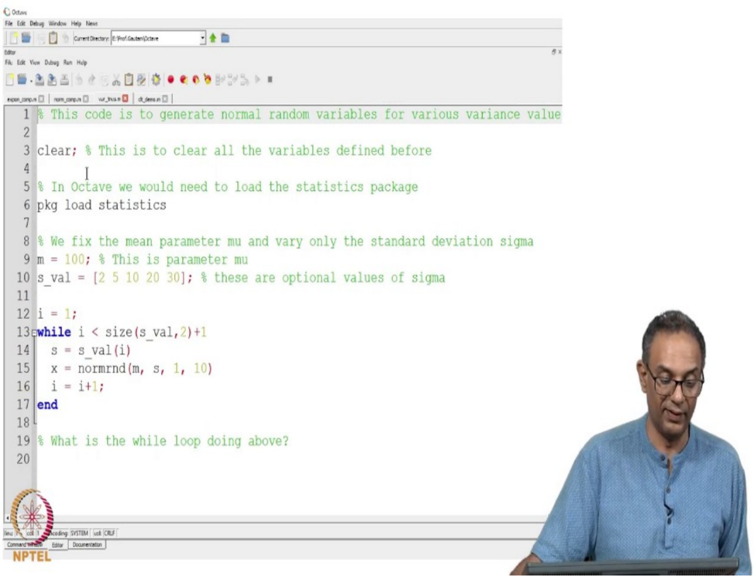

Now, the third one is what is called variance trials. So, it is called var_tries.m. Now, what I am going to do is, I am going to generate normal random variables and I am going to change the variance values and as I change the variance values, we are going to see how the numbers are generated and how varying they are.

So again, we are going to load the statistics package. I am going to do this a lot in this course. I am going to pick a mean of 100. So, we are going to a fix that. I am not going to mess with the mean; however, I am going to try five different standard deviation values – 2, which is very small in comparison to 100, 50, 10, 20, 30. So, I am going on increasing the standard deviation. So, these are the optional values that are used for σ.

So, what I am going to do is, I am writing a little while loop. Well, this is not the best way to do this. You would normally vectorize something like. The reason I am showing this is, it is much easier to explain. So, I am using this method; nothing wrong with it. But, anyone writing a code would probably do it more efficiently than what is presented here. So, whether i equals 1; that means, I am going to pick the first standard deviation, then I will go to the second, third, fourth and fifth.

So, I say while i is smaller than the size of this. So, if you want to play around with this and put a much larger value, let's say you go from 1 to 100 or something like that and see how the numbers look like, you are welcome to do so and then, it will pick the appropriate size of the vector. So now, this is less than; I go until that plus 1. So, basically what it does is right now is, it tells me that less than 6; so, it goes from 1, 2, 3, 4 and 5 and it uses that value 2, 5, 10, 20 and 30 as the standard deviation and computes normal random variables. It generates random normal random variables with this mean and this standard deviation. It generates exactly 10 of them. So, we can see them on the screen and it keeps doing this one by one, i equals i+1, it does till 5 and then, it stops. Now, notice this is what the while loop is doing about; it just goes on incrementing this value. So, the first time it picks 2, then it picks 3, then it picks 10, then it picks 20, then it picks 30. So, let's see how this works out; so, this is called var_tries.

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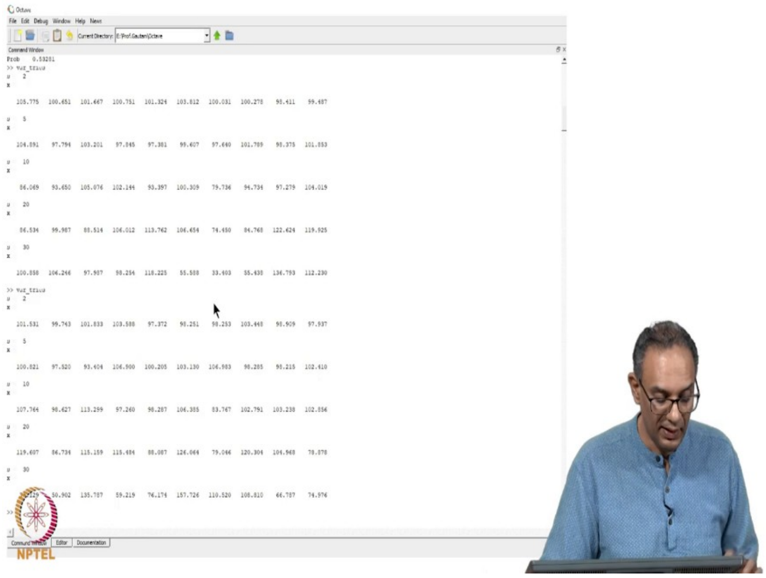

So, let’s do var_tries. Now, these numbers might be a little small; you have to squint a little bit. I do not find an easy way to increase the size of this. So, notice that when the standard deviation is 2, the numbers are all huddled around 100 which is the mean; the largest is probably 105 and the smallest is 98 something. And, when s is 5, when the standard deviation becomes larger, notice how the numbers are now they go a little bit more away from 100. Now, the numbers are all the way from 97 to 104. I realized that you randomly got a larger number here, but typically you see that range here is increased. When s equals 10, the variability goes even more; there are numbers from 79 all the way to 105. And then, when you pick 20, you see the number starting to get even wider and you start from 74. In this case, the specific example, I tried another set. Let me just choose one more time; you tried this again.

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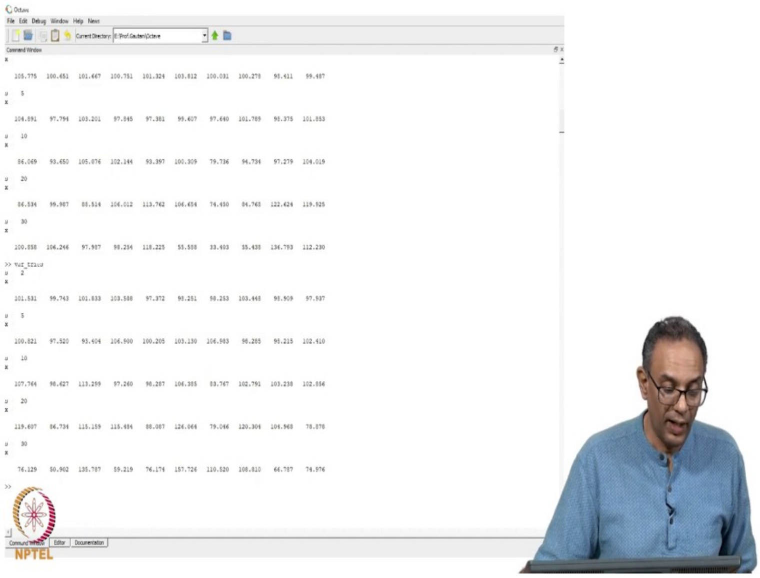

You will see that you get a different set than we did before. So, you look at it here. When s is 2, now the numbers are going from 97 to 103. When s is 5, the numbers are going all the way to 106 something, and as small. So, every time it gives you 10 different random numbers sampled from that distribution.

And, notice how the numbers are getting wider, farther away from 100 as the standard deviation increases. When s is 20, you see numbers that are close to 120 and numbers are small as 79. So, when it's 30, again you would see numbers that are over 130; so, 135 and as small as 59. So, the variability kind of goes up. So, as the variance goes up, the numbers become farther away from 100. That is what I wanted to make sure, that is, that these things are clear at your end, alright. So, I am going back to this, the presentation.

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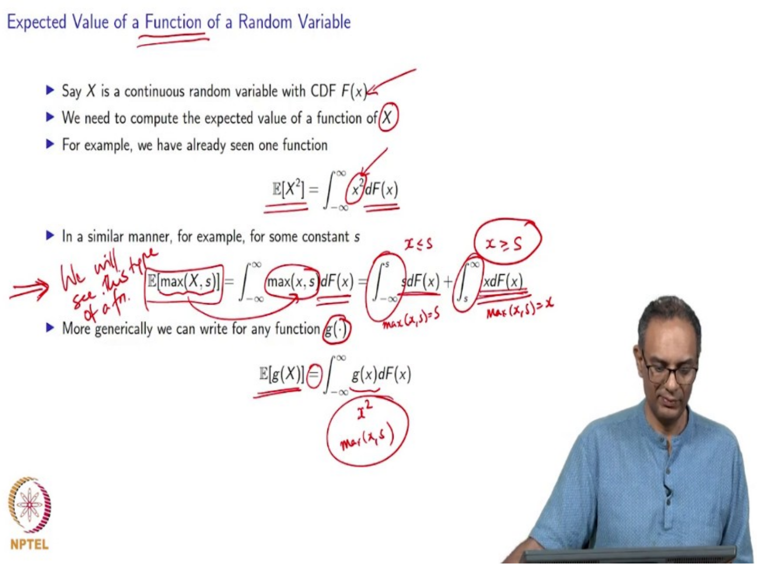

So, now let's go on and the last item in this particular topic is the expected value of a function of a random variable. So far, we just saw the expected value of a random variable X and then, we also saw one special function. So, let's come back to the continuous random variable with CDF $F(x)$; we want to compute the expected value of some function of X and I will give you an example.

So far, we have seen this $E\left[X^2\right]$. So, this is not the expected value of X, but the expected value of X^2. So, the way we compute that is, $E\left[X^2\right]=\int_{-\infty}^{\infty} x^2\, dF(x)$. In general, this X^2 could be any other function; let me give you another example. We will see this kind of a function a lot. We will see this type of a function in this course which is, $E\left[max(X,s)\right]$.

So, basically what you do is, $E\left[max(X,s)\right]=\int_{-\infty}^{\infty} max(x,s)\, dF(x)$. So, this part, $dF(x)$, always stays the same. This is interesting because the maximum of x and s changes when x is less than or equal to s and x is greater than or equal to s and all these values. So, $\int_{-\infty}^{\infty} max(x,s) dF(x)=\int_{-\infty}^{s} s\, dF(x)+\int_{s}^{\infty} x\, dF(x)$, since when $x \le s, max(x,s)=s$ and when $x \ge s, max(x,s)=x$. So, I can nicely decompose it this way.

More generically, if I want to compute the expected value of a function, $E[g(X)]$, g is any function, I could write that down as integral, $E[g(X)]=\int_{-\infty}^{\infty} g(x)dF(x)$. So, we did exactly the same thing this; instead of computing.; so, we saw X^2, we saw maximum of X and s; these are two examples. I could have any other function, $g(x)$. I plug it in here, $\int_{-\infty}^{\infty} g(x)dF(x)$, will give you the value. Now, we would not use a lot of this in this course except for this one particular guy and that's the reason I decided to use this, alright. So, this brings us to the end of this topic.

Thank you.