Decision Making Under Uncertainty
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Lecture – 04
Probability: Events, Conditioning and Total Probability

The next topic is probability and this will be somewhat mathematical. I would recommend taking a little bit of notes in terms of writing on a sheet of paper or something like that. So, we are going to be talking about probability of events, conditional probability and the topic of total probability.

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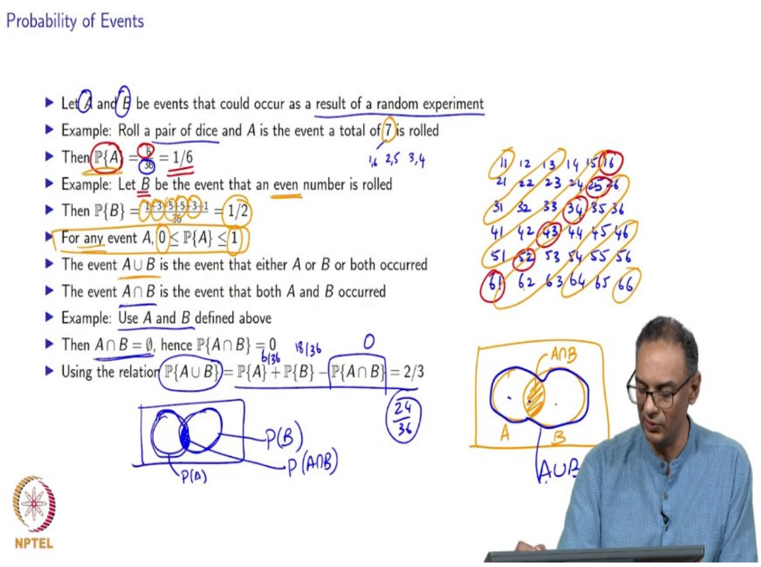

Say there are two events A and B. A and B are two events and these could occur as a result of a random experiment. Let me give you an example. It is easier to explain this using an example. The example is somewhat elementary and my approach is to start with something that is elementary and then take things up a notch as we go ahead.

Let’s say you roll a pair of dice. That means there are two dice that you roll and A is the event that you roll the number 7. So, what are different ways of rolling a 7? Well, you could roll a 7, by rolling either a 1 and a 6 or you could roll a 2 and a 5 or you could roll a 3 and a 4. However, when we count the probabilities, we want to be a little careful. Notice that there are 36 total possible outcomes.

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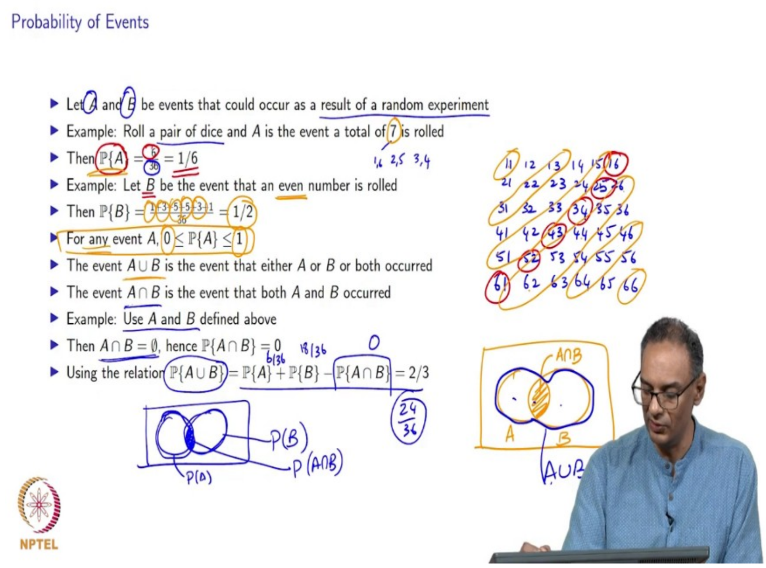

So, if you write it down, you could get a 1 1, 1 2, 1 3, 1 4, 1 5, 1 6 or you could get a 2 1, 2 2, 2 3, 2 4, 2 5, 2 6 and so on; 3 1, 3 2, 3 3, 3 4, 3 5 and 3 6; 4 1, 4 2, 4 3, 4 4, 4 5 and 4 6; 5 1, 5 2, 5 3, 5 4, 5 5 and 5 6; and lastly, you could get 61 through 66; these are the 36 equally likely outcomes.

Out of these 36 equally likely outcomes, if you look at this 1 6 whose sum is a 7; 2 5 whose sum is a 7; 3 4 whose sum is a 7; 4 3 whose sum is a 7; 5 2 whose sum is a 7; 6 1 whose sum is a 7. So, there 1 2 3 4 5 6 in total and that is the numerator 6; out of the 36 possible outcomes, 6 result in 7. So, the probability of rolling a 7 is 6/36 which is 1/6.

Now let’s look at another event. Say B is the event that an even number is rolled. So, we are looking at the total being an even number. Well, if you think about it, there are multiple ways of doing this. We could either directly count this; so essentially, it will be 1 1, then 3 of these - 3 1, 2 2, 1 3, then 5 of these - 5 1, 4 2, 3 3, 2 4, 1 5, then 5 of these 6 2, 5 3, 4 4, 3 5, 2 6, and then 3 of these - 6 4, 5 5, 4 6, and then 6 6. So, these are the various combinations that will result in an even number. The probability of rolling an even number is 18 out of 36; the numerator is 18 because it is equal to 1+3+5+5+3+1. 18 divided by 36 is equal to half. Well, this should not be surprising because you could either get an even number or an odd number, both of them being equally likely.

Therefore, the probabilities of the various numbers are not the same. So, you should not think just because it's like heads or tails; there is a certain difference there. However, if you look at this, this is an important piece of information for those of us that play games or board games.

So, the largest probability is probability of rolling a 7 which is 1/6; the probability of rolling a 6 or an 8 is 5/36 and so on.

One of the important rules of probability is that the probability of an event A should always be a number between 0 and 1. So, the most important rule of probability is that for any event A, the probability of any A is always a number between 0 and 1.

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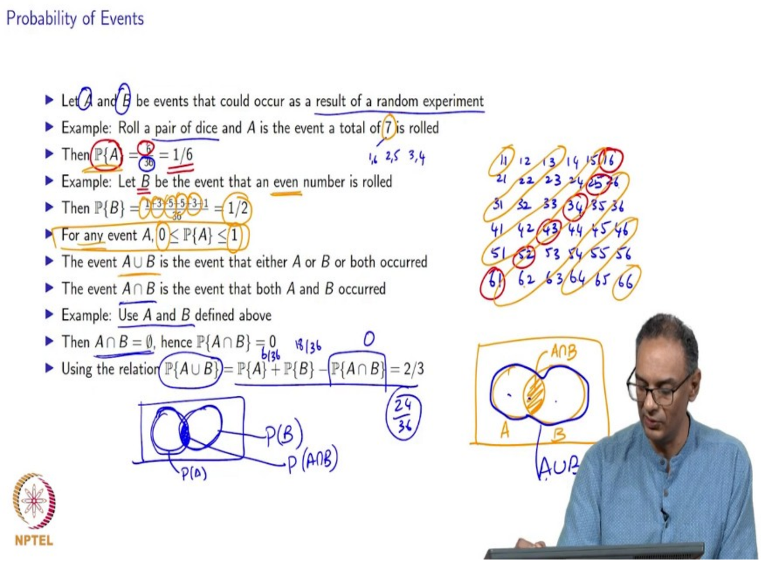

Next, we talk about events that are combinations of multiple events. So, let's use a Venn diagram. I am sure you have all seen Venn diagrams. So, let's say there is set A, there is set B, the space where both A and B occurs is A intersection B ($A \cap B$), and this entire set that I am going to use a different color - this entire set in blue is A union B ($A \cup B$). So, the event $A \cup B$ is event that either A occurs or B occurs or an event that is considered both A and B occurs. However, the event $A \cap B$ is the event that both A and B have occurred. So, let's use the A and B that we have defined above; that means, A is the event of rolling a 7 and B is the event of rolling an even number. Now, what is the probability of $A \cup B$? So, this is the probability of rolling either an even number or the number 7. If you look at it, in our events, $A \cap B$ is the null set because 7 is an odd number. So, the probability of $A \cap B$ is equal to 0.

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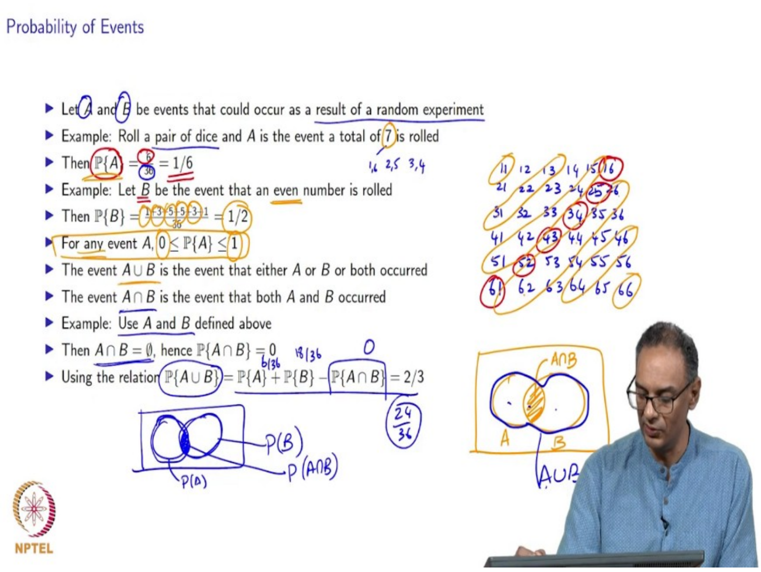

However, I want to talk about the more generic relation between A and B. So, the space where A occurs is called probability of A and the space where B occurs called is probability of B; however, $A \cap B$ is double counted. So, we have this result, the probability of $A \cup B$ is equal to the probability of A plus the probability of B minus the probability of $A \cap B$, that is, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. The reason we subtract it is because we have double counted the $A \cap B$ part; so, we subtract that.

Now, in this example, that is anyway 0. So, it's not that big a deal. However, in general, the probability of getting 7 is 6/36 and the probability of getting an even number is 18/36. Therefore, this is 24/36 which is two-thirds. That's how you get this probability of event. So, this is pretty straightforward; there is nothing tremendously complicated here.

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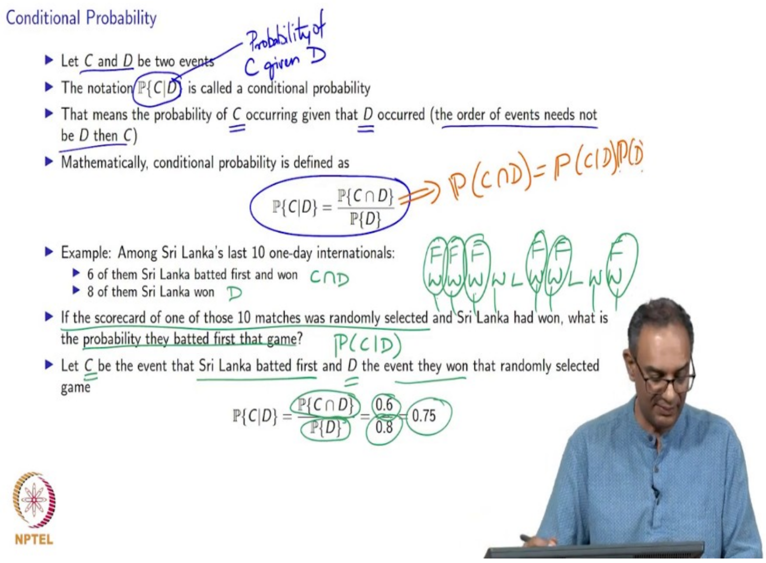

Next, we move on to the topic of conditional probability. Now, we are going to look at two other events C and D. We write this notation, $P(C|D)$, which means probability of C given D. We say this as probability of C given D; so, it is called conditional probability. It's conditioned on the value of D being given. So, some people also say this as the probability of C occurring given that D has occurred.

There are some very subtle things like the order of the events which are really not crucial; however, it's easy to understand it that way that D has occurred and then, C has not. How do we define this mathematically? So, mathematically we define the conditional probability in this fashion - the probability of $C \vee D$ is equal to the probability of $C \cap D$, that is both the events C and D occur, divided by the probability that event D occurs, $P(C|D) = \frac{P(C \cap D)}{P(D)}$.

So now, I am going to give you an example. In this example, we are going to look at the performance of Sri Lankan Cricket Team in the last 10 one day internationals. Let's say in those 10 one day internationals, Sri Lanka batted first and won in 6 games. So, we have two events. C is the event that Sri Lanka batted first and D is event that they won. The set $C \cap D$ denotes that they both batted at first and won. Now, 8 of these games, Sri Lanka won; so, this is your event D. Now, I am going to give you an example of how this could happen in the last 10 games where Sri Lanka won 8 games.

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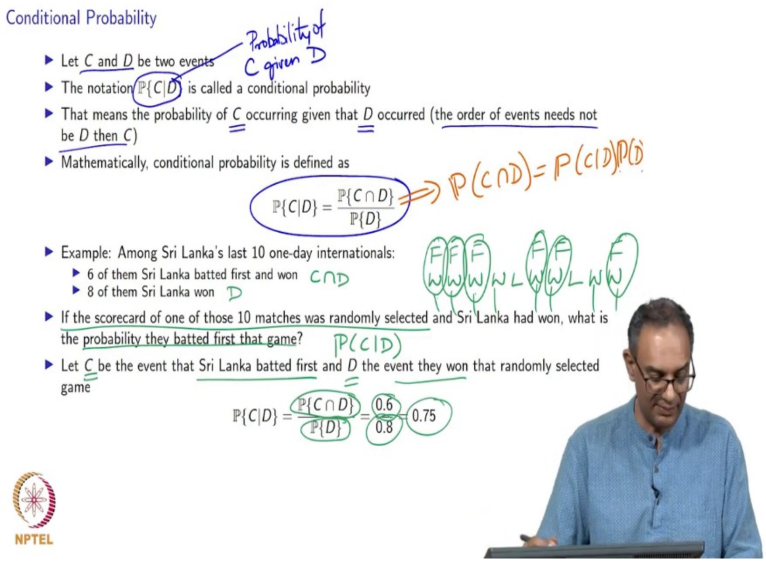

So, if you look at the last 10 games, Sri Lanka could have maybe won 4, then lost 1, then won 2, then lost 1 and then won 2 (W W W W L W W L W W). Out of these 8 wins, they batted first and won 6 times. So, they batted first in 6 of these games and won.

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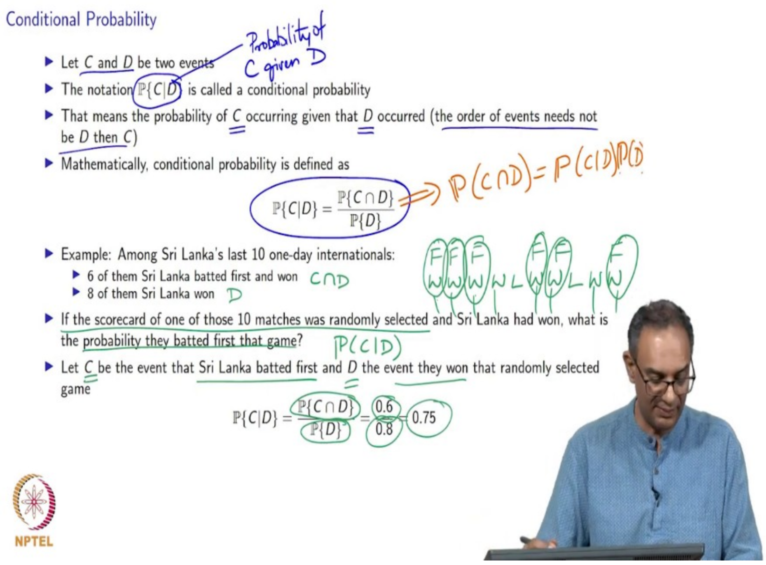

They batted first and won in 6 of these games. So, the 6 games where they batted first and won are 1 2 3 6 7 10. Clearly, we are in a situation where we have the data from the last 10 games of which there were 6 games in which they both batted first and won, and they won 8 games in total. Now, if we are saying - well, if I were randomly selecting one of these 10 matches, somebody randomly picked one of these 10 matches and someone told you, “Well, Sri Lanka won”. What is the probability that they batted first? So, we are in this same situation of probability of C given D because we are looking at the event C which is the event that Sri Lanka batted first; that’s what we want we are asking - what is the probability that they batted first? We don’t know that because we pick one of these games at random. We do know that Sri Lanka won. So, it’s one of these games that they won and I have asked you the question – “What is the probability that they batted first among those games?”. Well, clearly 6 out of those 8 games they batted first; that’s what this conditional probability essentially is telling you. So, it is the probability of $C \cap D$ which is 0.6 because 6 out of the 10 games they both batted first and won, divided by the probability of D which is 0.6. Therefore, $P(C \vee D)$ is 0.6/0.8 which is 0.75. Now, we look at it there. Among these 8 games that they won, they batted first on 6 of them. Therefore, you get 6 divided by 8 which is also 75 percent.

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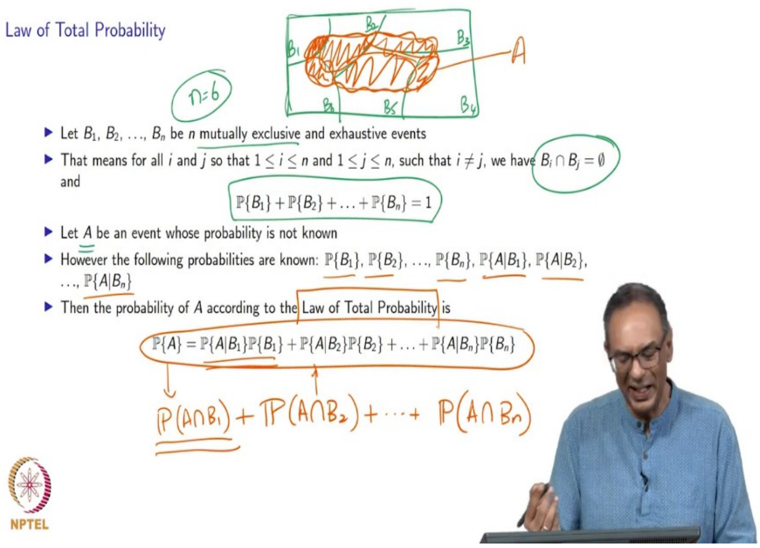

Now, let’s move along and talk about the next topic called the law of total probability. Now let’s say we have n different events, B_1 through B_n. I am going to represent these events using a Venn diagram. So, I give a specific example of n equals 1, 2, 3, 4, 5, 6, okay. So, I take the example of n equals 6. Say this is event B_1, this is event B_2, this is event B_3, this is event B_4, this is event B_5 and this is event B_6. Notice the characteristic of these events - they are mutually exclusive. By mutually exclusive, what we mean is that there is nothing common between two events. So, essentially $B_i \cap B_j$ is a null set where $i \neq j$; that means, $B_5 \cap B_6$ for example, is a null set because there is nothing common in them. So, that is mutually exclusive.

Also, we want exhaustive. So, the probability that either B_1 or B_2 or B_3 or B_4 or B_5 or B_6 has to happen; nothing else can happen. So, their probabilities add up to 1; they are exhaustive and exhaust all the possible outcomes of this experiment. So, the random experiment that could result in a bunch of events B_1 through B_6; n equals 6 in this example. And, those events are such that the events are themselves mutually exclusive and exhaustive; that means, if one event occurs, the other does not occur, and one of these n events must occur.

Now, I have another event called A. Now, this event A is one such that we don't know it's probability. So, I have an event A whose probability is unknown. We are asking the question - what is the probability of A? However, I do know not only the probabilities of B, but I also know these probabilities, the probability of A given B_1, the probability of A given B_2 and so on till A given B_n. I will give you an example to illustrate that in the next slide.

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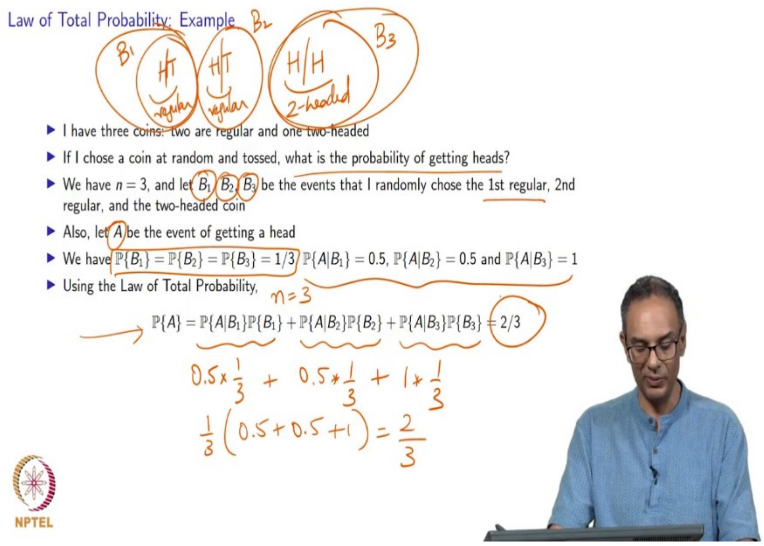

However, the result is pretty straightforward. So, if you think about it, this is your A. A is nothing, but $A \cap B_1$ plus $A \cap B_2$ plus $A \cap B_3$ plus $A \cap B_4$ plus $A \cap B_5$ plus $A \cap B_6$. So, we could typically write A, probability of A as the probability of $A \cap B_1$ plus the probability of $A \cap B_2$and so on, all the way to the probability of $A \cap B_n$.

However, from the previous result, we know that we can write down probability of $C \cap D$as the product of probability of $C \vee D$ times the probability of D. We can write it as a product and that is precisely what we will do here and get this result. We can write probability of $A \cap B_1$ as the probability of $A \vee B_1$ times the probability of B_1. So thereby, we can compute the probability of A using this and this is what is called the law of total probability.

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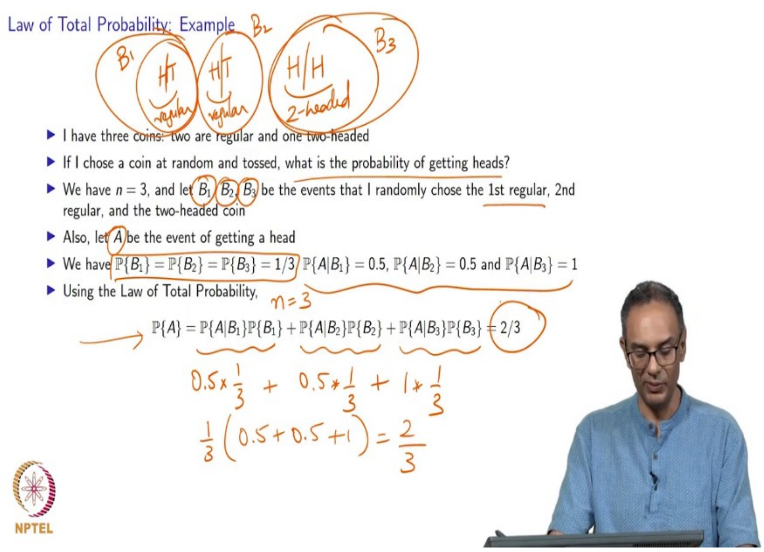

So, let's do an example to illustrate this concept. Say I have three coins. Two are regular- so, two coins are ones which will result in heads or tails; the third coin is 2 headed - this is a diabolical coin where I could get only heads. So, I have three coins where two coins are regular and one is a 2 headed coin.

Now, I am going to pick one of these three coins at random and I am going to toss it. The question is - what is the probability that I will get heads? Now, let's think of this in terms of the law of total probability that we just saw a little while ago. Let's say I have three events B_1 , B_2 and B_3. These are the events - the event of choosing the first coin is B_1, the event of choosing the second coin is B_2, and the event of choosing the third coin is B_3. These are my three events that could occur.

Notice that they are mutually exclusive, that is, I only pick one coin - it has to either be the 1st regular coin or the 2nd regular coin or the 3rd coin. It is also exhaustive, that means, if I pick the first regular coin, I did not pick the 2 headed coin. And, A is the event of getting a head. So, I don't know that probability right. Notice that I don't know the probability of getting a head, but I do know this. I do know two things - I know that the probability that I would select the regular coin 1 or the regular coin 2 are both equal to 1/3. However, the probability that I will select the 2 headed coin is also one third. They are all equally likely.

Now, what's the probability of tossing a head given that I pick the 1st coin or 2nd coin or 3rd coin? If I pick the first regular coin, the probability of scoring a head is half because I have head or tails in the regular coin. Now, if I toss the second coin, again I have the option of getting a head or tail; so the probability is half. However, in the 3rd coin, I will always get a head with probability 1 for sure.

Therefore, I have these three probabilities that are quite easy to compute like I said a little while ago. Now, if you use the law of total probability and write this down in the following way, remember n equals 3 here; the probability of A is the probability of $A \vee B_1$ times the probability of B_1, plus the probability of $A \vee B_2$times the probability of B_2, plus the probability of $A \vee B_3$ times the probability of B_3. If you write down the numbers, $A \vee B_1$ from here is 0.5 multiplied by the probability of B 1 is 1/3, plus 0.5 times 1/3, plus 1 times 1/3.

So, if you add them up, the probability of getting a heads is 2/3. So, it is somewhat straightforward because in reality, there are 6 different heads that could have showed up right when you toss a coin. So, after the 6 faces that could show up - the 2 tails and the 4 heads, what is the probability that 1 of those 4 head shows up? So, you could have also computed that from basics and said, "Well, that has to be 2/3 because there are 4 heads out of these 6 faces that could have showed up and therefore, the probability that you will get a head is 4 out of 6 which is 2/3". Alright. Next, we move on to random variables which will be the next topic.

Thank you.