Decision Making Under Uncertainty
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Lecture – 31
DTMC Modeling and Analysis

The next topic of our lecture is called DTMC Modeling and Analysis.

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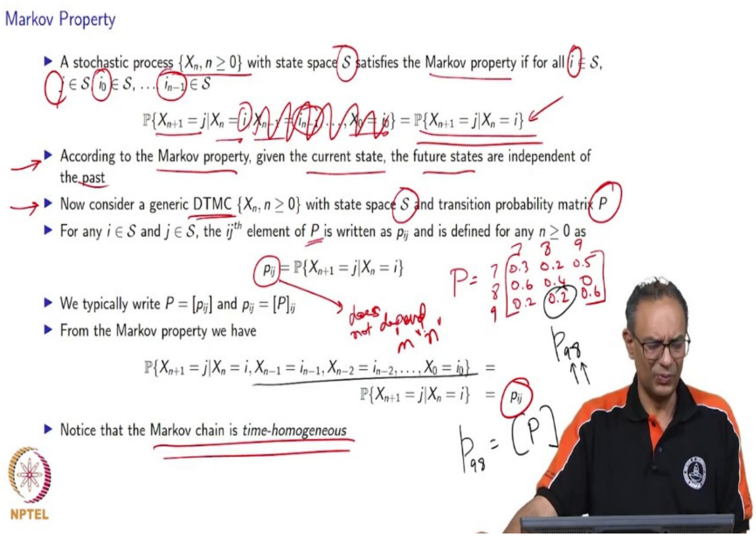


So, we modeled a system as a discrete time Markov chain. We said the word Markov property, I am going to mathematically state what it is formally. And then, we will go about analyzing the Markov chain. So, a stochastic process X_n which state space S satisfies the Markov property if we have the following results.

I am going to take a bunch of states i which is part of the state space S; j which is part of the state space S. i_0, i_1, i_2,…., i_{n-1}. I am going to show you why. Now, this is where the conditional probability comes in. The probability that the n +1st observation is in state j given; so I know my entire history until this point from time 0 all the way till the nth observation, I know what states I have been in. I was in state i_0, then I went to i_1, i_2 and so on.

I was in i_{n-1}, then now I am in state i, this is my current observation, I have my entire history. My next observation is going to be in state j with this probability. Now, turns out all this history is irrelevant is what the Markov property says. I do not care how I got the state i. So, I am right now in state i, I do not care how I got there. Next time I will be in state j with probability given by this. Now, think about the washing machine example.

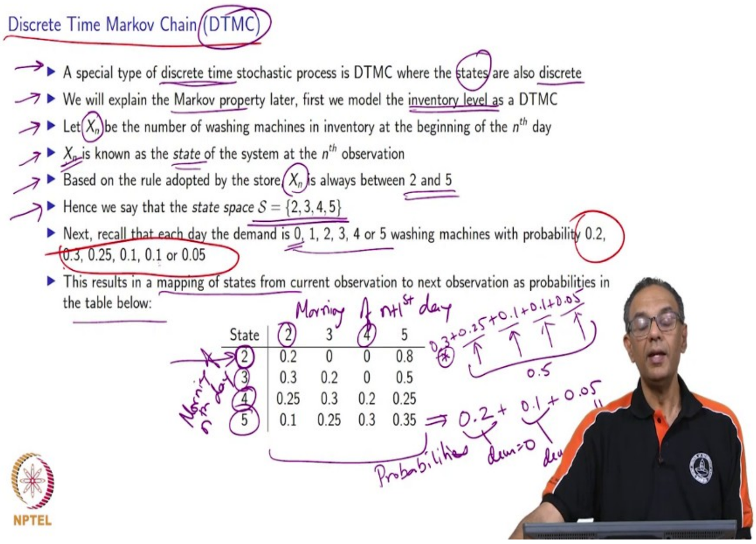
So, let us say right now I have three items in inventory, I might know what would have happened in the previous time. So, I do not even need it. In order for me to predict what will happen in the next time unit, all I need to know is where I am right now. And if you think about it, you could write your X_{n+1} itself in terms of your X_n. So, the n^{th} observation- how many ever I had and then whatever random demand I got, that I can use to predict how many I will have the next time.

So, I do not really need to know what happened in the past. So, in word the Markov property is as follows: given the present or the current state, the future is independent of the past. The future is independent of the past or whatever happened so far, as long as I know my current state, I do not need to know how I got there. I can predict the future.

Now, let us pick a generic discrete time Markov chain X_n which has a state space S and transition probability matrix P. So, let us just pick one of those. Now, typically we write this thing p ij as the ij th element of the P matrix right. So, this is the i j th element of the P matrix. So, usually this p_{ij} is essentially what we have in our matrix. So, for example, if P is; I am going to give you another example let us say this is 7, 8, 9, 7, 8, 9.

So, let us say this is 0.3, 0.2, 0.5, 0.6, 0.4, 0, 0.2, 0.2, 0.6 have a discrete time Markov chain like this. Then, this particular element is called p_{98}, so it is called p_{ij}. There, first number tells me my current states, the second number tells me my next state that is why it is called p_{ij}. We sometimes write p_{98} as the matrix P is element corresponding to row with state 9 and column with state 8.

It is not really the 9^{th} row and the 8^{th} column like we sometimes write down when we have matrix operations. Now, another thing you have to remember is: why is those two the same thing, while the same Markov property gives us this result. So, this p_{ij} that we have is essentially the p_{ij} that we saw earlier, is essentially the ij^{th} element of the Markov chain. Notice that this Markov chain is what we call time homogeneous and the reason it is time homogeneous is because these probabilities that we had before do not vary with time. (Refer Slide Time: 04:29)



So, since the probabilities are exactly the same day in and day out. These numbers do not change from day to day, then the Markov chain is also stationary or does not vary with time. So, that is an important point- it is not time varying, so homogeneous over time. So, it does not matter when you observed it, the probability of going from state i to state j is p_{ij}. So, this guy p_{ij} does not depend on n, it is not time dependent, so it is time homogeneous. It is just a minor technicality nothing to worry about.

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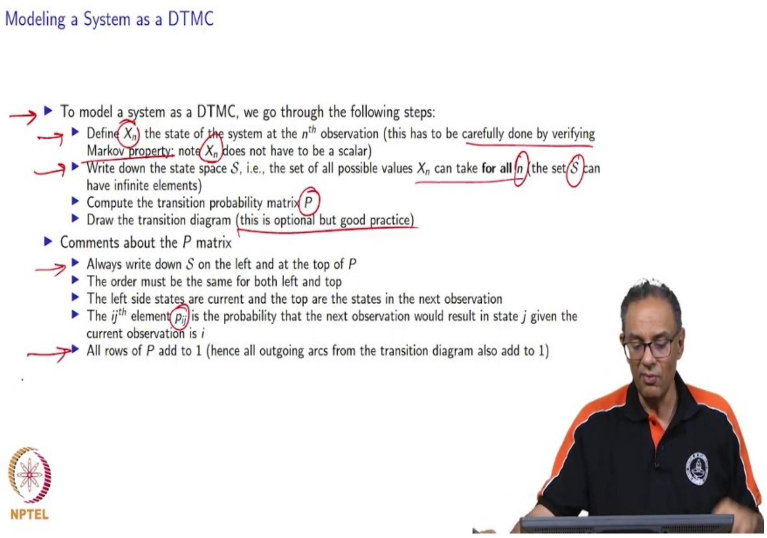

Now, let us see how we will model the system as a discrete time Markov chain. There are a few steps to follow, in a discrete time Markov chain, the first thing that you need to do is define X_n, which is the state of the system at the n^{th} observation. So, what are you going to observe, like I was telling you let us first decide when to observe, I am not even putting that in there, like in the inventory problem.

You decide observe the beginning of the day and X_n is how much inventory I have. Now, what you choose to observe, you have to be careful because if you do not do a good job of that, you might not have Markov property satisfied. So, it is important for you to think about what we shall observe; in this problem it is very straightforward, but other times it could be difficult. So, you have to pick something that would guarantee that your system is going to have Markov property. The second thing is X_n does not have to be a scalar- it does not have to be a single number.

For example, I could observe the inventory level of 10 items and have a 10 dimensional vector for X_n. So, I could think of like you have 10 brands of washing machine in my floor. And each brand of washing machine, I can tell you how many items that are in the state.

Why is that important? Well, if you had limited space, then you can have more of one and less than the other and you can kind of need to adjust. And when you place an order, you place an order for all of them. You do not just place an order for one of them and things like that. The next thing to do after writing down X_n is to figure out what are all the possible values that X_n can take: this is my state space.

Now, please remember that S is only one thing, it is not defined for every n. So, this S is all the possible values that all X_n can take. So, for every possible X_n from 0 to infinity- when an n equals 0 to all the way through n equals infinite, over all possible values it can take that is your S. Then, you can compute the transition probability matrix P, which we did and then draw the transition diagram.

Now, drawing the transition diagram is an optional thing, but I would recommend doing that because it is good practice and we will see how we use that when we do the analysis. There are few comments I would like to make about the P matrix. Now, you should always remember to write down the S: the state space on the left and the top. Always remember the left is the current state, the top is the next state.

The order must be the same for both the left and the top. Then the left side states are the current states, the top states like I said are the next observation. Then the p_{ij} element remember is a probability after all. So, if given that I am in state i, what is the probability that i will be in state j. So, because of that all the rows and P have to add to 1 right. So, for example, if right now I am in state i, I have to be in one of the other states. So, the probabilities must add to 1. I am going to say all this using an example.

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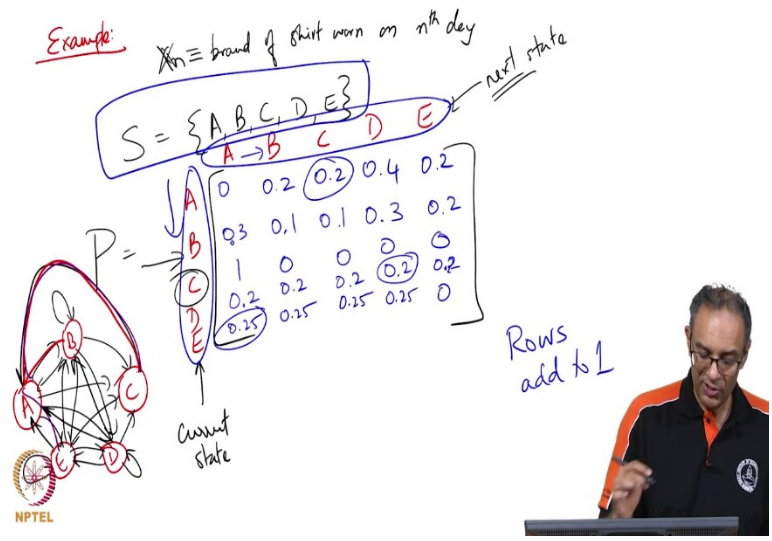

So, let us do another example, in this example what I am going to do is: I am going to say X n, X n is let us say the brand of shirt worn on the n^{th} day. So, let us say you generally have 5 brands of shirt, then your state space is let us call those brands A, B, C, D and E. These are 5 brands of shirts and you have a transition probability matrix like this. Now, this is my A, B, C, D and E; A, B, C, D and E.

Now, notice the following things. First these are my current states. So, this is the brand of shirt I am wearing today, this is the next state- this is the brand of shirt I will wear tomorrow. So, this is what is given, this is the next state is what we have next.

So, now for example, this could be 0, this could be 0.2, this could be 0.2, this could be 0.4, this could be 0.2. Notice that my rows at the 1 because if I wear a “A” type of shirt today, tomorrow I have to be wearing a “B or C or D or E or A”. Something, one of those 5, I have to wear. Because that is my state space, those are all the possible values that I can take, that the type of shirts that I could be wearing. So, therefore, those numbers will have to add to 1. So, 0.2 + 0.2 + 0.4 + 0.2 = 1.

So, from B for example, you could wear A the next day, maybe you wear B again, then you or you can wear C, or you can well D or you can wear E. So, the rows add to 1. Likewise, in the third; so, let us say you started today and stay shirt C. Whenever I wear C, next day I will wear A, like a 1 in there if I wanted to; there is nothing that stops me from having a 1 in there.

Let us say I wear a D type of shirt, then the next day I will wear anything, but a D. So, let us just say: we will wear each one of them with probability 0.2 equally likely. But I am in E, I will wear anything, but an E with probability quarter. So, notice how my rows add to 1, so notice the rows add to 1.

Notice that I have written A, B, C, D, E in the same order top to bottom, left to right is in the same order, which is also the order that I write down my S my state space. They are all in the same order that is very important and I also have made sure that my rows add to 1. So, if you look at the previous slide these are them always write down S on the left and top, order must be the same and then the left are the current states and the top is the next state, p_{ij} is the probability that the next observation will be in state j given that current is i and the rows add to 1.

So, for example, this number tells me: if today I am wearing a D, tomorrow also I will wear a D with probability 0.2. If I am wearing an A today, then tomorrow I will wear a C with probability 0.2. If I am wearing an E today, tomorrow I will wear an A with probability 0.25; this is what is on these numbers.

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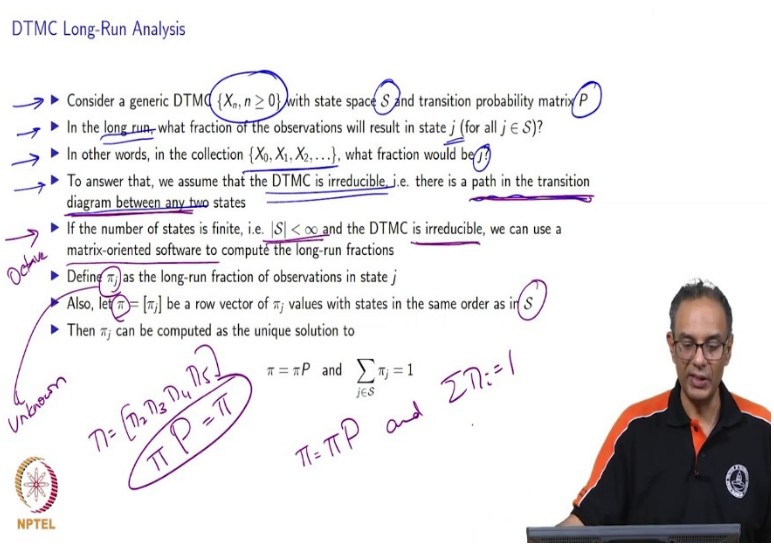

Now, let us do what is called long run analysis. There is something called transient analysis that I have skipped. In a usual Markov chains course, we would normally do a transient analysis and I am skipping that here. I do want to let you know that in case you look up somewhere and the next thing that they talk about is transient analysis. And you wonder what why did I skip it, that is because we are mainly going to be focusing on what is called long run analysis or steady state analysis.

Now, I am going to pick a generic discrete time Markov chain X_n, by that what I mean is I am not sticking with a specific example, I will do that in the next slide. It has a state space S and transition probability matrix P, the question: is in the long run what fraction of observations will be in state j.

So, for example, the inventor examples, some of the times some of the fraction of observations you will observe system in 2, some of them in 3, some of them in 4, some of them in 5. What fraction of observations you think you will be observing in the long run in state 4. So, if you want to ask that very generically.

So, you see this series right X_0, then X_1, then X_2. These are random, they depend on the previous value and so on. And what fraction of these will be in this state j for some j in the state space. For that, we need a condition- we are going to assume that the discrete time Markov chain is irreducible. That means, there is a path between any two states.

So, for example, let us take the previous one itself. If I were to draw a transition diagram for that; I am going to go back to my red and then I am going to do A, B, C, D, E. From A, I can go to everything except A. From B, I can go to every state including itself. So, that is not a problem. From C, I can only go to A.

So, draw an arrow like that. From D, I can go to every state and from E, I can go to everything, but itself. So, this is a highly well connected graph. Now for example, from C, I can only go to A, but I could eventually get to B, that is what I want to show. From C, I can go to A. From A, I can get to B. For example, there is no path let us say from C to E, then I can go from C to A and from A, I can go to E.

So, there is a path in the transition diagram and that is one of the reasons you must draw a transition diagram between every two states. Pick a state there is a path. There may not be a direct arc from every state to every other state, but there is a path. So, if the number of states is finite; now we will not get into the situation where the cardinality of S is infinite; that means an infinite state.

We will look at the case it is finite. When it is finite and the Markov chain is irreducible, we could immediately use some matrix oriented software. We are going to use octave like we have done throughout this course to compute this long run fraction. I am going to show you a little demo of how one goes about doing this in octave. So, now we are going to let π be the long run fraction of observations in stage j. So, π_j is a probability that the long run fraction of observations is in stage j.

So, we do not know this, this is an unknown quantity. We do not know this, we wish to compute it. So, let π be a vector a row vector, so π is equal to for example, in our inventory example would be a row vector of π_2, π_3, π_4 and π_5.

So, this is basically all the values of π in the state space S. Now, turns out this π satisfies the equation: $\pi = \pi P$ and $\Sigma \pi_i = 1$. So, that means, if you multiply π by the P matrix, you get back π. And $\Sigma \pi_i$, that means, $\pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$. So, I am going to illustrate this using an example.

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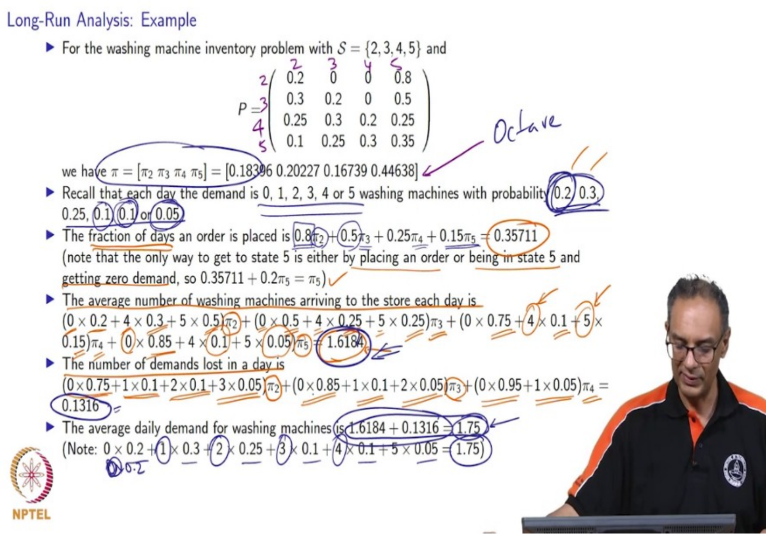

So, we take our old example again and then 2, 3, 4 and 5; 2, 3, 4 and 5. So, this is our P matrix. Now, we could show that π is this and I am going to use a little software to actually do that, but before that I am going to show you how one goes about doing that.

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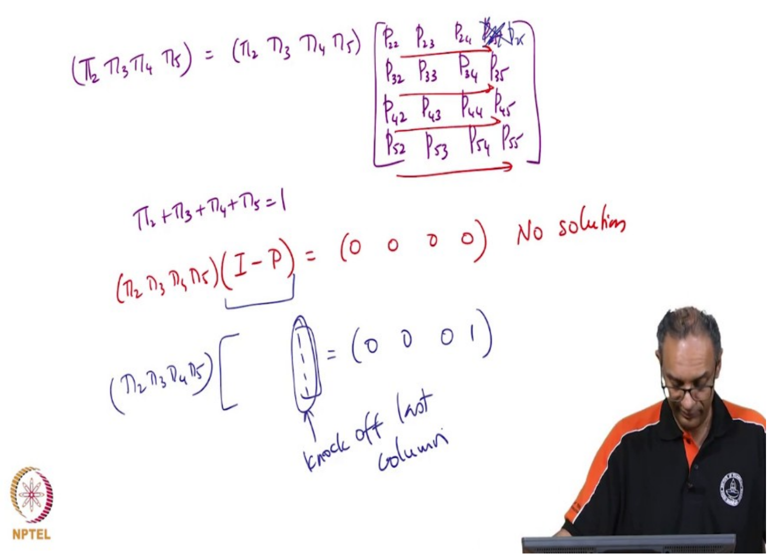

So you try to solve for $\pi = \pi P$. So, that means, you would write $\pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$ equals ($\pi_2 + \pi_3 + \pi_4 + \pi_5$¿ X P. So, I am going to it for now write P_{22}, P_{23}, P_{24}, P_{25}, P_{32}, P_{33}. I know I actually have the numbers, but I want to illustrate something in particular 42, 43, 44 and 45 and P_{52}, P_{53}, P_{54}, and P_{55}; this is true and also write $\pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$.

So, we want to solve for this equation, how I go about doing that. So, one of the tricks that we adopt is- if you try and solve for this equation by writing down $\pi_2, \pi_3, \pi_4, \pi_5$times the identity matrix minus P equals 0 0 0 0 most software will not invert this for you. Therefore, you will get in no solution, this will give you no solution.

One way is to write down 5 equations with 4 unknowns. Now, turns out that because these add to one, the equations that you have are not going to be linearly independent. One of the equations is already dependent. So, you can really throw one of the equations out.

So, for that what I am going to do is, I am going to say: ok, I am going to instead of solving for this equation, what I am going to do is: I am going to replace this oh sorry let me just leave this as P 25. What I am going to do is instead of having: $I-P=0$, I am going to take the last column and make this as 1 and convert this $\pi_2, \pi_3, \pi_4, \pi_5$ and convert this as 0 0 0 1. So, 0 0 0 1 and I knock off the last, so this is knock off last column. Now, I can solve this using a little matrix solver, so I am going to show that to in octave.

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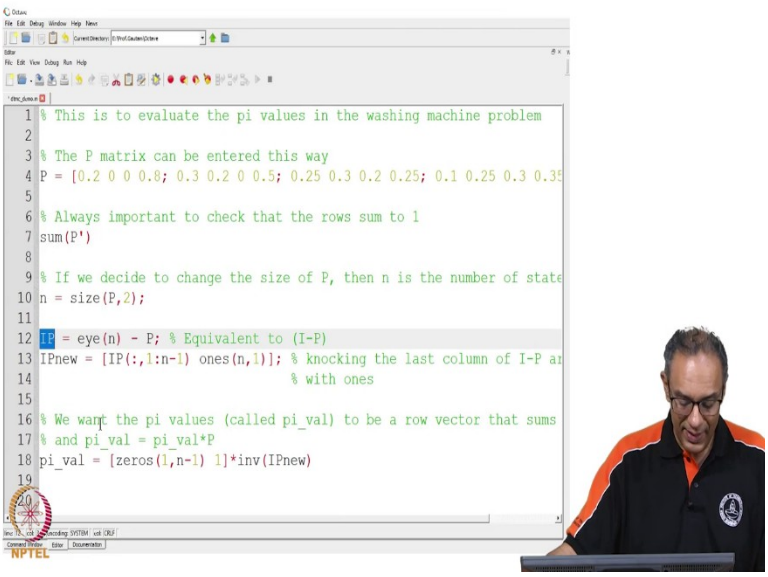

So, let us look at the octave program. So, this one is to essentially evaluate the π for the washing machine problem. Now, you could change the P matrix for another problem. Now, if you notice here the P matrix is not fully visible. So, I am going to bring it back here.

So, it is not too bad. So, this is 0.2: the probability of going from 2 to 2, this is 0, this is 0, this is 0.8. And then, put that semicolon to say that I am going to the next line. Then 0.3, 0.2 is 0 and 0.5. Then you go to the third line, this is probability going from 2, 3, 4 to the various states and then probably going from 5 to the various states, so that is a P matrix.

I always check if the sum of the P', remember in octave if you do sum, it will actually do column sums. So, I do P' to get the row sums because it gives you the transpose of the matrix and computes it. Now, in general I am going to write n as the size of the P matrix, I am going to a size(P,2). In this case we know n = 4, but what if I you want to go ahead and play with this by trying a different P matrix.

Then it will pick n, then what I do is this I P is this I - P that we had here this; I - P that we have here in this part right here. We are going to write call that as the letter I P or the letters I P is I this eye is the identity matrix of n. So, it tells me that it is 4 dimensional minus P.

Now, this is I minus P, but I want to knock off the last column. So, I take the first n - 1 columns, right all rows and first n - 1 column, leave them the way they are. And the last column, I am going to replace by ones and which is going to get a bunch of 1s. This is exactly what we have in the last column. Last column is knocked off and we replace by 1s. And the first n - 1 columns stay the way they are. Now, we want the π values.

So, essentially what I do is: I take the inverse of this new matrix I P new and multiplied by 0s n - 1 0s and then a 1, so that is the operation. If you are not very sure, take your time to go through this code and make sure that it runs well. So, we are going to do DTMC underscore demo. So, if you look at this- it first tells me the row sums are equal to 1. So, that is good and the π values are given by this: 0.18 0.2, 0.16 and 0.44, these values are listed in here.

So, if you look at it here, this is 0.18, this is what we got from octave 0.18, 0.2, 0.16, 0.44, so that is what we got as your π. So, this is how we calculated it, you can do this for any finite value P matrix: not finite value, I mean finite sized P matrix. By the way, you could have infinite size too and nothing stops you from that.

Now, remember that each days demand is 0, 1, 2, 3, 4 or 5 with these probabilities, we know that. And now we are asking the question: what fraction of a days do we place an order? So, think about the following, at the beginning of a day if you had two items, then the probability of placing an order is 0.8.

Why is that the case? Well, there is a 0.8 chance that I will be down to 1 or 0 right, anything, but 0.2. If I had 0 demand, then I won't. 1 - 0.2 = 0.8, is a probability that the next that I would have placed an order the end of that day. Likewise, if I am state 3, there is a 0.5 chance that I would have placed an order, and that is because that is everything, but having 0 or 1 demand. Likewise, if I am in state 4, let us have 4 items in my in inventory at the beginning of a day, there is a 25% chance that I will place an order.

And that is because either I had a demand of 5 or 4, or 3, I would place an order. Likewise, if I came with 5; now this you have to be a little careful. If I came down with 5, I had demand of 5, I would place an order. I had a demand of 4, I would place an order, so 0.15. So if you multiplied those numbers, you get this nice value this 0.35711: this is the fraction of days that you place an order. So, little more than a thirds of the days, you place an order.

Now, it is important to note the following- note this result. If we take this point, this is the number of days you place an order. 0.35711 you add to that. The probability that you will be in state 5 and place an order, that would happen if you are in state 5 and with probability right, this is not the only way to get 5. So, when will you get to 5: either you place an order or you are in state 5 and you get 0 demands that is the only way to be in state 5.

So, either you place a new order or you are in state 5 and get a 0 demand with probability 0.2 that should be the long run fraction of time you observe the system in state 5. Now, you could do a quick calculation and see that this equation is actually satisfied. I am not doing that here, but you are welcome to try that using octave.

Now, another result that I want to explain is that if you compute the average number of washing machines that arrive to the store each day. So, if I had 2 in the system with probability 0.2, I will have 0 arriving. With probability 0.3, I will have 4 arriving because 0.3 is the probability that I will go down to 1 washing machine. So, I only need to order 4. And 0.5 is the probability that I will be down to 0, that is because I have a demand anything, but 0 or 1 which is 0.2+0.3=0.5. So, everything else I will order 5 of them.

Likewise, if I am in state 3 with probability 0.5, I will order 0; with probability 0.25, I will order 4; with probability 0.25, I will order 5. And then, in state 4, again you are always going to order 0, 4 or 5. With probability 0.75, I will order 0 that is because when you are in state 4 the only time you will order is if you had a demand of either 5 or demand of 4 or a demand of

3. So, that happens with probability 0.1 and 0.15; I should not use the word respectively, you know when you order 5, this is when your demand is either equal to 5 or 4.

This is if your demand is equal to 3. Likewise, if you under state 5, you would place an order for 0 with probability 0.85, this means you know I got a demand of either 0, 1 or 2, or 3 or that is it 3. If you had a demand of 4, then you would place an order for 4; if your demand of 5, then you place an order for 5. If you did the calculations this will be the average number of washing machines arriving to the store each day, the number of demands lost each day.

So, you can do whole bunch of analysis like this- how many demands will you lose each day? So, if you are in state 2 and you had a demand of 0 or 1, you are not going to lose anything, or a demand of 2 you are not going to lose anything. However, a demand of 3, you would mean 1 person you would lose; a demand of 4, you would mean 2 people you lose; a demand of 5 would mean you lose 3 people.

Likewise, if you are in state 3, well then with demand of 0, 1, 2; you will lose and 3 you will lose nothing; demand of 4 you will lose 1; a demand of 5, you will lose 2. Likewise, in state 4, a demand of 5, you will lose 1; a demand of anything other than 5, you will lose nothing. And in state 5, you will not lose anybody. Now, it is important to realize that there are two numbers that are crucial. Here, the first number is this guy and the second number is this guy. This is the average number that arrive into a store, these are the average customers that get rejected.

So, the daily demand for washing machines if you think about it is equal to all the machines that they arrived per day- the average number that arrived per day and the average number of customers that were lost per day- is the average demand for washing machines. Everything that came, either the customers came and went sad without a washing machine or they were delivered from the store, this is going to be your demand. Now, I could compute the demand in a different way, this is the standard expected value of demand, 0.2 times the probability that.

So, 0 is if I had a demand of 0; 1 is if I had a demand of 1; 2, if I demand of 2; 3 a demand of 3 and a probabilities at 0.2, 0.3, 0.25, 0.1, 0.1 and 0.25. If I did the computations they are exactly equal. So, and they must be equal right because all your demand is either going to be satisfied by new washing machines or they are not going to be sad or you know or somebody comes and not get a washing machine. So, these two together must add to your average demand. That brings us to the end of this presentation.

Thank you.