Introduction to Operations Research
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Module - 02
Graphical and Algebraic Methods
Lecture - 03
Algebraic Method (Maximization)

In this class, we will study the Algebraic Method to solve linear programming problems. We will begin by considering a Maximization Problem subject to less than or equal to constraints. We will look at the same example that we used to illustrate the graphical method.

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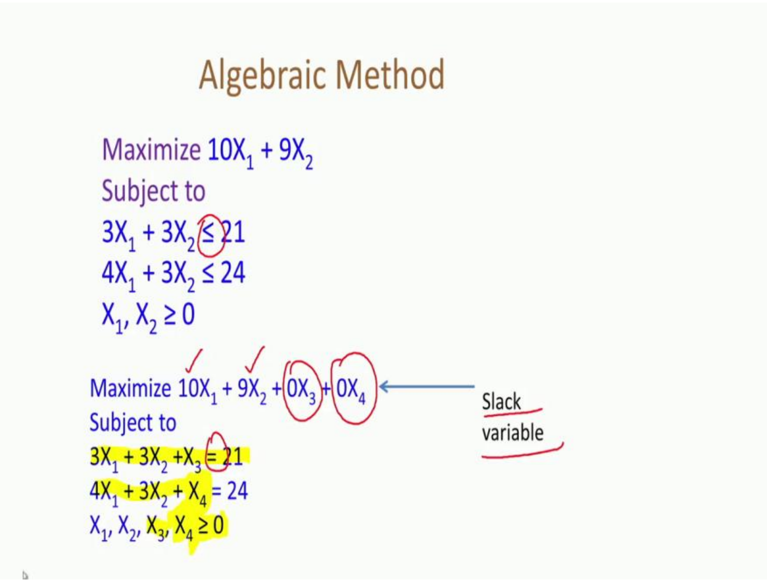

So, we go back to the same maximization problem which maximizes 10 X 1 plus 9 X 2 subject to 3 X 1 plus 3 X 2 less than or equal to 21 and 4 X 1 plus 3 X 2 less than or equal to 24, X 1, X 2 greater than or equal to 0. Now, in order to solve this algebraically let us first look only at the constraints. For a moment let us set aside the objective function and for a moment let us set aside the non-negativity. So, we are looking at 3 X 1 plus 3 X 2 less than or equal to 21 and 4 X 1 plus 3 X 2 less than or equal to 24. So, we have two inequalities.

So, at the moment let us again assume that we know to solve equations while we do not know at the moment to solve inequalities. So, under the assumption that we know to solve equations let us see how we can apply that idea to solving the two given inequalities. So, in order to solve them as equations, the first thing is to write them as equations from inequalities. So, we have 3 X 1 plus 3 X 2 less than or equal to 21 and X 1, X 2 greater than or equal to 0.

So, if we have to satisfy X 1, X 2 greater than or equal to 0 and 3 X 1 plus 3 X 2 less than or equal to 21. It is now possible to introduce another variable called X 3 and write 3 X 1 plus 3 X 2 less than or equal to 21 as 3 X 1 plus 3 X 2 plus X 3 equal to 21 which is what we have shown here, we have written 3 X 1 plus 3 X 2 plus X 3 equal to 21. Now, since X 1 and X 2 they have to be greater than or equal to 0 and they have to satisfy 3 X 1 plus 3 X 2 less than or equal to 21.

Therefore, 3 X 1 plus 3 X 2 can either be equal to 21, in which case X 3 will be equal to 0 or if 3 X 1 plus 3 X 2 is less than 21, then X 3 will take a positive value. Therefore, X 3 will now be defined as greater than or equal to 0, so x 3 is greater than or equal to 0. So, 3 X 1 plus 3 X 2 less than or equal to 21 is now rewritten as 3 X 1 plus 3 X 2 plus X 3 equal to 21. It is written as 3 X 1 plus 3 X 2 plus X 3 equal to 21 and X 3 greater than or equal to 0.

In a similar manner 4 X 1 plus 3 X 2 less than or equal to 24 is written as 4 X 1 plus 3 X 2. So, it is written as 4 X 1 plus 3 X 2 plus X 4 equal to 24 and X 4 will be greater than or equal to 0. Now, X 4 is greater than or equal to 0, because if 4 X 1 plus 3 X 2 is equal to 24 then X 4 will become 0. If 4 X 1 plus 3 X 2 is less than 24, then X 4 will take a positive value therefore, X 4 is greater than or equal to 0.

So, the given problem maximize 10 X 1 plus 9 X 2 subject to 3 X 1 plus 3 X 2 less than equal to 21, 4 X 1 plus 3 X 2 less than equal to 24, X 1, X 2 greater than or equal to 0 is now written as maximize 10 X 1 plus 9 X 2 plus 0 X 3 plus 0 X 4 subject to 3 X 1 plus 3 X 2 plus X 3 equal to 21, 4 X 1 plus 3 X 2 plus X 4 equal to 24, X 1, X 2, X 3, X 4 greater than or equal to 0. Now, these variables X 3 and X 4 that are written here are called slack variables.

Now, these slack variables are used to convert an inequality into an equation. So, this inequality is converted to an equation by the addition of slack variables. Now, when we formulated the problem we said the revenue associated with X 1 is 10, the revenue associated with X 2 is 9. So, the revenue or profit or money that is associated with X 3 and X 4 are 0 and it is assumed that they do not contribute to the objective function.

Now, we have converted a problem with inequalities into a problem with equations and we now try to solve this problem. Now, we started by saying that, since we now to solve equations, we will now look at these two equations and try to solve them. So, we have two equations, but now with the addition of the two slack variables, we now have four variables X 1, X 2, X 3 and X 4. When we learnt methods to solve linear equations, we learnt methods where the number of variables that we solve is equal to the number of equations.

But, now in our example we have four variables and we have two equations, now when we have two equations, we can solve only for two variables. Now, we have to find out a way by which we solve for two variables at a time when we have four variables and two equations.

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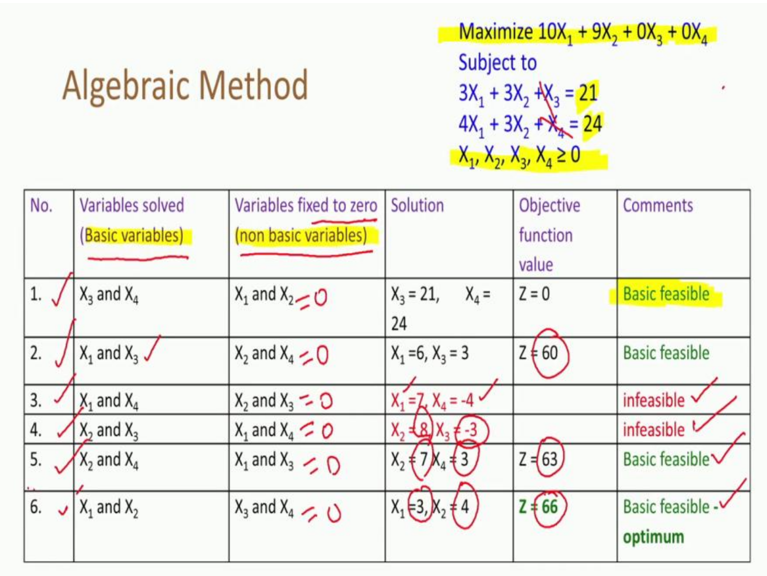

No.	Variables solved (Basic variables)	Variables fixed to zero (non basic variables)	Solution	Objective function value	Comments
1.	X_3 and X_4	X_1 and X_2	$X_3 = 21$, $X_4 = 24$	Z = 0	Basic feasible
2.	X_1 and X_3	X_2 and X_4	$X_1 = 6$, $X_3 = 3$	Z = 60	Basic feasible
3.	X_1 and X_4	X_2 and X_3	$X_1 = 7$, $X_4 = -4$		infeasible
4.	X_2 and X_3	X_1 and X_4	$X_2 = 8$, $X_3 = -3$		infeasible
5.	X_2 and X_4	X_1 and X_3	$X_2 = 7$, $X_4 = 3$	Z = 63	Basic feasible
6.	X_1 and X_2	X_3 and X_4	$X_1 = 3$, $X_2 = 4$	**Z = 66**	Basic feasible - **optimum**

So, let us now see how we solve this, so the problem that we have now or we are now going to solve is given on the right hand side. The problem that we are going to solve is given here maximize 10 X 1 plus 9 X 2 plus 0 X 3 plus 0 X 4 subject to the first equation here, second equation here and so on. So, there are four variables and two equations and since we have two equations we can solve only for two variables at a time.

So, we choose two variables that we are going to solve, which means the other two variables that we are not going to solve, we have to give them some value. For example, if we assume that we are going to solve for X 1 and X 2 we need to give some value for X 3 and X 4. If we give a value of 1 for X 3 and X 4, then the equations will become 3 X 1 plus 3 X 2 equal to 20 and 4 X 1 plus 3 X 2 equal to 23. If we give 0 value for X 3 and X 4 it will become 3 X 1 plus 3 X 2 equal to 21 and 4 X 1 plus 3 X 2 equal to 24.

So, we have to give some value for the two variables that we are not going to solve which are called fixed variables, we have to give them some value. Now, we can give any value to X 3 and X 4 except that we want X 3 and X 4 to be greater than or equal to 0. Therefore, fixing X 3 and X 4 or fixing any two variables we can do it in infinite possible ways, the easiest thing to do and perhaps the first thing we can try and do is to fix these variables at 0.

So, what we are going to do is, at any point we are going to solve for two variables by fixing two other variables to 0. Those variables that we are going to solve are called basic variables and those variables that we are going to fix to 0, we are not going to solve for these they are called non basic variables. Now, we have a total of four variables in this formulation that we have on the right hand corner. So, we have four variables in this formulation that we have there are four variables at any point I am going to solve only for two variables and I am going to fix some other two variables to 0.

So, if I am going to solve for two variables out of 4 the two variables that I am going to solve can be chosen in 4 C 2 ways or they can be chosen in 6 ways. So, they can be chosen in 4 C 2 equal to 6 ways I can choose the variables. Now, these 6 ways by which I can choose them are written here from 1 to 6. So, one pair could be I am solving for X 3 and X 4 which means I am fixing X 1, X 2 to 0.

I can solve for X 1, X 3 by fixing X 2 and X 4 to 0, I can solve for X 1 and X 4 by fixing X 2, X 3 to 0. I can solve for X 2 and X 3 by fixing X 1 X 4 to 0, I can solve for X 2 and X 4 by fixing X 1, X 3 to 0 and I can solve for X 1, X 2 by fixing X 3, X 4 to 0. So, I can solve six problems I can choose two variables out of 4 in 6 ways 4 C 2 ways that have been shorn in this corner and I can solve the problem using these 6 pairs variables.

In each of these six problems there are two basic variables, the variables that we are going to solve and we have the remaining two variables as non basic variables and we solve we fix the non basic variables to 0. So, those variables that are fixed to 0 are the non basic variables and those variables that we are going to solve are the basic variables. Now, let us look at each one of these six solutions and see what happens when we actually solve them.

The first out of the six solutions we are going to solve for X 3 and X 4 and we are going to fix X 1 and X 2 at 0. So, when we solve for X 3 and X 4 we are fixing X 1 and X 2 to 0, so I am just striking X 1, X 2 off now we can easily see that the solution is X 3 equal to 21, X 4 equal to 24. Because, X 1 X 2 are fixed at 0 they are not going to contribute. So, we will have X 3 equal to 21, X 4 equal to 24 that is shown here as the first solution.

Now, we move to the second solution coming back to the first one we said we fixed X 1, X 2 to 0 therefore, we got the solution X 3 equal to 21, X 4 equal to 24 which is written here X 3 equal to 21, X 4 equal to 24. We now evaluate the objective function value for this, because this solution is basic feasible. So, let me explain what a basic feasible solution is, now this solution is basic because we have fixed X 1, X 2 2 out of the four variables to 0 and more importantly we are solving for two variables X 3 and X 4.

So, we are solving for X 3 and X 4 those variables are called basic variables, non basic variables are fixed at 0 we get a solution X 3 equal to 21, X 4 equal to 24, X 1 equal to 0 X 2 equal to 0. Now, this solution X 1 equal to 0, X 2 equal to 0, X 3 equal to 21, X 4 equal to 24 satisfies the non negativity restriction of X 1, X 2, X 3, X 4 greater than or equal to 0 and also satisfies these two equations which are now the constraints. Because, this solution satisfies all the constraint it becomes feasible and since it is a basic solution it becomes basic feasible.

So, again a solution is basic when we solve for as many variables as the number of equations, there are two equations, two constraints we solve for two variables the basic variables here are X 3 and X 4. The non basic variables are set to 0, X 1 and X 2 are set to 0. So, when we solve we get a solution X 3 equal to 21, X 4 equal to 24, now this is a basic solution. And since this basic solution is also a feasible, it satisfies this constraint this and the non-negativity restriction, it becomes a basic feasible solution.

Once we have a basic feasible solution, we find the value of the objective function, the objective function is 0 because both X 1 and X 2 are 0, X 3 and X 4 do not contribute to the objective function. So, the value is 10 into 0 plus 9 into 0 which is 0, we now move to the second of the six solutions. Now, in this second solution we are going to solve for X 1 and X 3 by keeping X 2 and X 4 to 0. Since, I am keeping X 2 and X 4 to 0 I am writing these off X 2 and X 4 are not here.

So, directly I get 4 X 1 equal to 24 from the second equation that gives me X 1 equal to 6 and once I substitute X 1 equal to 6, 3 X 1 is 18 therefore, X 3 is equal to 3. So, this basic solution where we are solving for two variables X 1 and X 3, fixing the remaining variables to 0 gives us X 1 equal to 6 and X 3 equal to 3, this is also a basic feasible. Because, X 2 equal to 0, X 4 equal to 0, X 1 equal to 6, X 3 equal to 3 satisfies the non negativity restriction and therefore, this is basic feasible.

Now, we evaluate the objective function value, so X 1 equal to 6, X 2 equal to 0 would give us 10 X 1 plus 9 X 2 equal to 10 into 6 plus 9 into 0 which is 60, so we get the second solution which is here. We now look at the third solution, in the third solution we are going to solve for X 1 and X 4, so when we solve for X 1 and X 4 we have X 2 and X 3 equal to 0. So, X 2 and X 3 are now kept equal to 0, so from the first equation we will get 3 X 1 equal to 21 we get X 1 equal to 7.

So, when we substitute X 1 equal to 7 in the second equation we get 28 plus X 4 equal to 24, X 4 is minus 4 and therefore, this solution is infeasible and does not satisfy the non negativity restriction. So, we write it as infeasible and we do not evaluate the objective function, we now go to the fourth one where we are going to solve for X 2 and X 3. So, we will eliminate X 1 and X 4 here. So, this would give us 3 X 2 equal to 24 and then we would get X 2 equal to 8 which is shown here.

Now, when we substitute X 2 equal to 8 in the first equation, we get 24 plus X 3 equal to 21 we get X 3 equal to minus 3 and this solution is infeasible, because it violates the non negativity restriction. We now move on to the fifth solution, where we solve for X 2 and X 4 and therefore, we will eliminate X 1 and X 3 from the first equation we get 3 X 2 equal to 21, so X 2 is equal to 7. Now, we substitute in the second equation 7 into 3 21 plus 3 equal to 24, X 4 equal to 3. So, this is basic feasible because it satisfies the non negativity restriction, the value of the objective function is 9 into 7 which is 63.

The six and the last one is we solve for X 1 and X 2 here by eliminating X 3 and X 4. So, we solve for X 1 and X 2 by eliminating X 3 and X 4 and when we solve for 3 X 1 plus 3 X 2 equal to 21, 4 X 1 plus 3 X 2 equal to 24 we get a solution X 1 equal to 3, X 2 equal to 4 which is also a basic feasible. So, we evaluate the objective function to get 66 which is shown here and 66 is the best solution that we have out of the basic feasible solutions and hence is the optimum solution to this problem. In the next class, we will revisit this algebraic method and we will try to get more insights into the algebraic method.