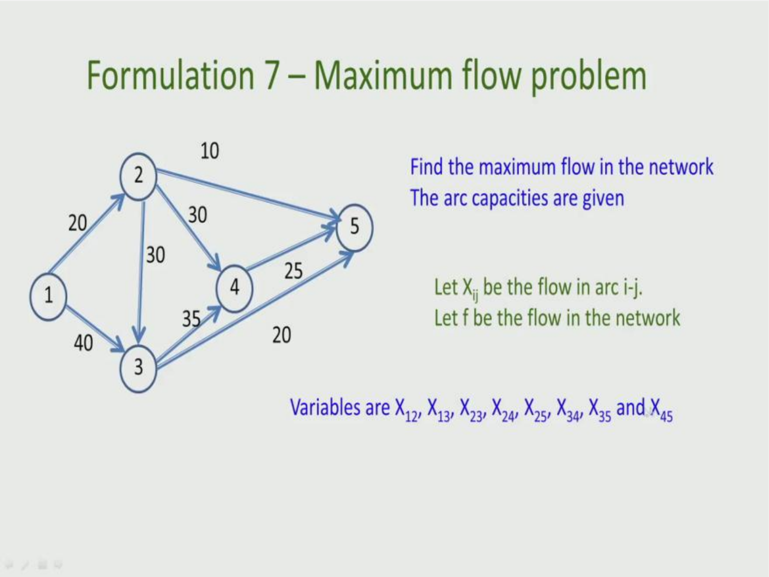
Introduction to Operations Research
Prof. G. Srinivasan
Department of Management Studies
Indian Institute of Technology, Madras

Module - 01
Linear Programming - Introduction and Formulations
Lecture - 05
Maximum Flow and Bin Packing Problems

In this class we will look at two formulations, the Maximum Flow Problem and the Bin Packing Problem, so our formulation 7 is the maximum flow problem. We are given a network and we want to find out the maximum flow in the given network, the arc capacities are given in the network.

(Refer Slide Time: 00:50)

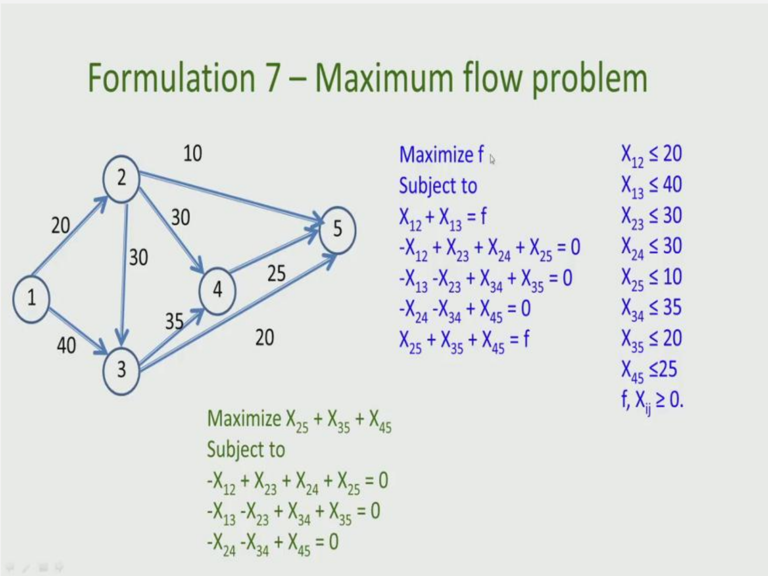


So, this is the network that we are going to consider. This network has 5 nodes or vertices, these are the nodes and the network has arcs connecting these nodes. So, these are this arc from 1 to 2, these arcs are represented as i j and they go from i to j. For example, 1 2, 1 to 3, 2 to 3 and so on, this network is a directed network which means arcs are given directions. Now, these arcs have capacities, we want to find out the maximum flow that can be sent through this network, which means we will assume that these points 1 and 5 will act as some kind of a source and destination.

Now, one is called the source, five is called the destination, sometimes five is also called the sink. So, some commodity or some fluid is flowing from 1 to 5 along these arcs which act like pipes, the arc capacity is equivalent of what can flow in this arc. So, with this we start defining the formulation for the maximum flow problem. So, let X i j be the amount of flow in the arc from i to j, so there are many arcs. So, there is an arc from 1 to 2 therefore, there is a X 1 2, there is an arc from 1 to 3 therefore, there is a X 1 3, there is no arc connecting 1 and 4, so there is no X 1 4.

So, we will have as many variables as the number of arcs in this network. So, let X i j be the flow in the arc i j or the arc that connects nodes i and j. We also define f as the flow in the network which we try to maximize. Now, let us write the objective function and the constraints for this problem.

(Refer Slide Time: 03:05)



So, we want to maximize the flow in this network, so the flow in this network is f, so we want to maximize f. So, maximizing f is the objective function, f is the variable that represents the flow. Now, what is the flow? The flow is something that flows out of the source or the flow is the same as that which flows into the sink or destination. So, let us consider vertex 1 or node 1 which acts as the source, so what goes out of 1 is X 1 2 plus X 1 3. So, X 1 2 plus X 1 3 goes out of 1 and that represents the flow.

So, the first constraint which is for node 1 is X 1 2 plus X 1 3 is equal to f, f is the flow. So, whatever flows is equal to X 1 2 plus X 1 3, now if we look at the sink or the destination which is 5. Now, we can look at the 5th constraint, now what enters the sink is X 2 5 plus X 4 5 plus X 3 5 and the same flow that went out of 1 will have to reach 5.

So, the constraint for 5 or destination or sink is given by X 2 5 plus X 3 5 plus X 4 5 is again equal to f. X 2 5 is coming this way, X 3 5 is coming here, X 4 5 is coming here, now the remaining three vertices which are 2, 3 and 4 act as intermediate vertices.

So, if I look at node 2 or vertex 2, whatever comes into node 2 will have to leave node 2. So, X 1 2 is the flow coming into node 2 and the flow that leaves node 2 are X 2 3, X 2 4 and X 2 5. So, the constraint will be X 1 2 should be equal to X 2 3 plus X 2 4 plus X 2 5 and that is shown as this constraint, which says minus X 1 2 plus X 2 3 plus X 2 4 plus X 2 5 equal to 0, which is the same as X 1 2 is equal to X 2 3 plus X 2 plus X 2 5.

Similarly, looking at node 3 whatever comes into 3 has to leave 3, so what comes into 3 is X 1 3 plus X 2 3. So, the flow that enters node 3 is X 1 3 plus X 2 3, the flow that leaves node 3 is X 3 4 plus X 3 5. Therefore, X 1 3 plus X 2 3 should be equal to X 3 4 plus X 3 5 which is written as minus X 1 3 minus X 2 3 plus X 3 4 plus X 3 5 equal to 0. Similarly, for node 4 whatever comes into node 4 should be equal to what leaves node 4. So, X 2 4 plus X 3 4 is equal to X 4 5 which is written as minus X 2 4 minus X 3 4 plus X 4 5 equal to 0.

So, there are five constraints corresponding to each of these nodes, the constraints corresponding to the source and the sink have the flow variable. The other constraints corresponding to the intermediate nodes do not have the flow variable and whatever comes into that node should leave that node or vertex, so five constraints are written. Now, in addition to these we have the capacity restrictions on the arcs, now X 1 2 is the flow in the arc connecting 1 and 2 and that cannot exceed 20 units therefore, X 1 2 is less than or equal to 20.

Similarly, X 1 3 is less than or equal to 40 and so on. So, there are 8 arcs and we have 8 capacity constraints which are given here in addition, the non-negativity constraints f and X i j are greater than or equal to 0. So, this completes the formulation of the maximum flow problem, the objective is to maximize the flow, the flow is given by f there are 5 constraints one for each node or one for each vertex and these are generally called as flow balance constraints, they are built on the principle that for every node whatever comes in has to go out.

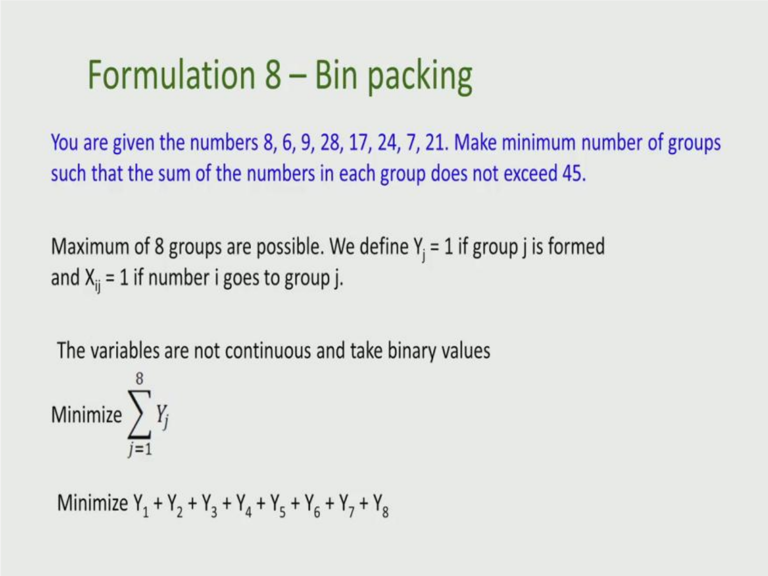
Now, if we assume that f comes into 1, then f will be equal to X 1 2 plus X 1 3. Similarly, if f goes out of node 5 then f will be equal to X 2 5 plus X 3 5 plus X 4 5. We also have the arc capacity constraints, there are as many constraints as the number of arcs and the non-negativity restriction. So, the number of variables is equal to the number of arcs plus 1, the one is for the flow variable, the number of constraints is equal to the number of nodes or vertices plus the number of arcs.

Now, we make a minor change in this formulation and that is shown here. For example, we can eliminate this f variable, now we have use this f explicitly as a variable that denotes the flow and we now know that this flow is equal to X 1 2 plus X 1 3 and the same flow is also equal to X 2 5 plus X 3 5 plus X 4 5. So, we can replace this f as maximize X 2 5 plus X 3 5 plus X 4 5 and then, we can write the constraints for the nodes 2, 3 and 4 and so on.

So, this way we will reduce the number of variables by 1 and we also reduce the 2 constraints corresponding to the flow so, but we retain the arc capacity restrictions. So, the new formulation will not have the f variable, it will have only the X i j variables. So, it will have as many constraints as the number of arcs which are the arc capacity constraints with respect to the flow conservation equations, we will use one of them as the objective function, we will not necessarily use the other, the other three for the intermediate nodes are actually sufficient.

So, this formulation will have as many variables as the number of arcs, it will have as many constraints as the number of arcs which are the flow conservation or the capacity constraints and with respect to flow conservation, we will have as many constraints as the number of intermediate nodes. We could either use the formulation that explicitly has f and there is a pattern there, the first way by which be formulated has X i j and f it has an extra variable which is f, it has as many constraints as the number of nodes plus as many constraints as the number of arcs, we could use either of the formulation. But, the second one where we have eliminated the flow variable has one variable less. So, this is how we formulate the maximum flow problem.

(Refer Slide Time: 11:49)



And we now move on to another formulation which is called the bin packing problem it is also called the one dimensional bin packing problem. So, let me introduce the problem in a slightly different way, now you are given 8 numbers and these numbers are 8, 6, 9, 28, 17, 24, 7, and 21. Now, the problem is to make minimum number of groups such that the sum of the numbers in each group does not exceed 45 or does not exceed a given number, so this is a grouping problem.

Now, we have to form minimum number of groups such that the sum of the elements or numbers in each group does not exceed 45, it is fairly obvious that we 8 groups we can easily have a solution to this problem. But, the real challenge is what is the minimum number of groups and which number goes to which group, such that the sum of elements in any group is less than or equal to 45, it is called a one dimensional bin packing problem, because these 8 numbers that I have given here can be treated as 8 lengths.

So, you could imagine sum 8 sticks of certain length, lengths are 8, 6, 9, 28 and so on and then you want to put them into a bag in a one dimensional way and the bag or the bin as it is called has a height of 45. So, when I put it in a one dimensional way I would not be able to put for example, 9 plus 28 plus 17 into a bin of 45 because it exceeds the bin length. So, in general the bin length is given, so in this example the bin length is 45 I want to pack these items or length into bins in a one dimensional way such that I use minimum number of bins and I am able to pack all these lengths or items.

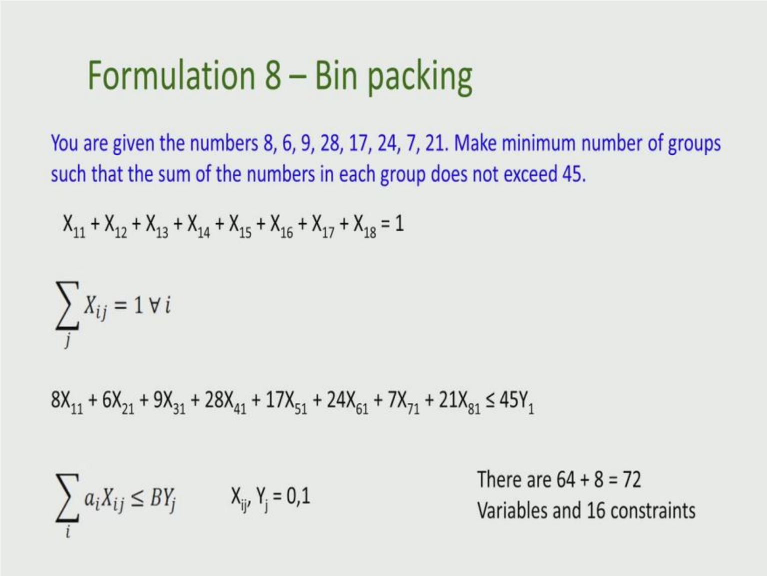
Therefore, it is called a bin packing problem, it is also called a one dimensional bin packing. Because, we are packing the bin in one dimension, which means whatever you put in the bin we have to add the lengths, which is the same as grouping them and finding out the sum of the elements in the group and that sum of elements in the group should not exceed the predefined number which is 45. So, let us try and formulate this problem.

Now, maximum of 8 groups are possible, so we first say let y j be equal to 1 if group j is formed and y j equal to 0, if group j is not formed. We also define X i j equal to 1 if the i'th number goes to group j and X i j equal to 0 if the i'th number does not go to group j. The first significant difference is that the variables are not continuous variables and these variables take only binary values and these variables take only values 0 or 1. Now, y j is 1 if group j is formed, y j is 0 if group j is not formed.

For example, if we do not have to form an 8th group then y 8 will be 0, if we have to form the second group then y 2 will be equal to 1. For example, if the first number 8 goes to the first group, then X 1 1 is equal to 1 and if it does not go it will be 0. So, we have in this formulation 8 variables for y j and 64 variables for x i j. Because, each number can go to any one of the 8 groups, so there are 64 binary variables in this formulation. Now, the objective function will be to minimize the number of group that we form.

So, minimize sigma Y j which is minimize Y 1 plus Y 2 plus Y 3 plus Y 4 up to Y 8, if we have to form 8 groups then all the 8 variables will take value 1, if we have to form 3 groups then 3 out of the 8 will take value 1. So, minimize sigma Y j is the objective function.

(Refer Slide Time: 16:40)



Now, the constraints are if I take any one of the words for example, the number 8 as the first number then this number or this element or this item should go to only one of the bins. Now, at the moment we are considering 8 possible bins, so X 1 1 plus X 1 2 plus X 1 3 up to X 1 8 equal to 1 means the first subscript one which is the first number one, the first number which is 8 will either go to the first group or the second group or the third group or the fourth group or up to the eighth group.

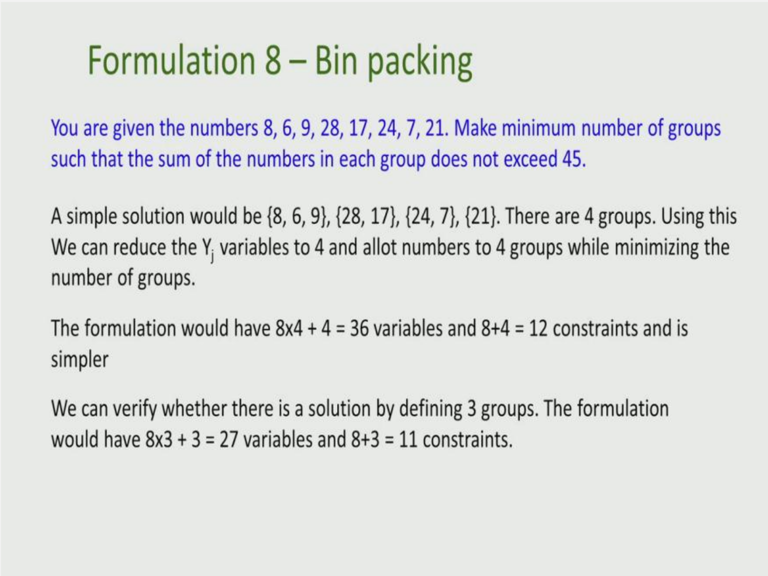
But, I will go to only one out of these eight groups, so sigma X i j equal to 1 summed over j for every i. So, there are 8 constraints like this, for the second number the constraint will look like X 2 1 plus X 2 2 plus X 2 3 plus X 2 4 plus X 2 5 plus X 2 6 plus X 2 7 plus X 2 8 equal to 1. So, like that for each of the 8 numbers there will be a constraint and there will be 8 constraints here. So, for every i there are 8 numbers there are 8 constraints summed over j X i j equal to 1 each number goes to only one out of the 8 possible groups.

Now, each group that is formed should have a total less than or equal to 45. So, if I am forming the first group then Y 1 is equal to 1, so the right hand side becomes 45. So, whatever goes into the first group which means if the first element goes then it is going to take away 8 out of the 45. If the second number goes it is going to use up 6 out of the 45. So, 8 X 1 1 plus 6 X 2 1 plus 9 X 3 1 plus 28 X 4 1, etcetera is whatever goes into the first group and if the first group is formed, then whatever goes into the first group the sum should be less than or equal to 45.

If the 8th group is not formed then Y 8 will become 0, so the right hand side will become 0 and no number will go to the 8th group. So, sigma over i a i X i j is less than or equal to B Y j is a very general way of representing two things, if a group is formed then whatever goes into that group should be less than or equal to 45, some of those elements should be less than equal to 45. If the group is not formed, then nothing goes into that group.

So, this is the way by which we formulate, so there are 72 variables, 64 for the X i j variables and 8 it for the Y j variables 72 variables and there are 8 plus 8 16 constraints which are here, this is for every element should go to one group, this is for every group when it is formed will have elements such that the sum is less than or equal to 45. So, this formulation has 72 variables and 16 constraints all the variables are binary variables. Now, let us do a little bit into this problem.

(Refer Slide Time: 20:17)



Let us make a simple solution by looking at these numbers and start putting these numbers into groups. So, as I start putting these numbers into groups I realize that 8, 6 and 9 can go to the first group, because if I add 28 it exceeds the 45. So, 28 and 17 will go to the second group, 24 and 7 will go to the third group and 21 will be the 4th group. So, I am able to get a solution with 4 groups, so I use this idea.

So, now, I can reduce the Y j variables to 4 and not 8, in the earlier formulation I had 8 possible groups. Now, I have a solution with 4 groups, now I want to check is their solution with 3 groups. So, I reduce the Y variables to 4 and allot the numbers only to 4 groups, while we minimize the number of groups and now such a formulation will have 8 into 4 plus 4 36 variables it will have only Y 1 plus Y 2 plus Y 3 plus Y 4 as the objective function, it will have only 12 constraints and it is a simpler formulation.

We can even go one step ahead and say that I already have a solution with 4 groups, now do I have a solution with 3 groups. So, now, I restrict Y only to Y 1 Y 2 and Y 3 which means the number of variables comes down to 27 variables and 11 constraints. So, this is a type of a problem where, if we start with the solution to the problem I can get a more efficient formulation. So, the bin packing formulation now has explained a few things.

Now, for the first time we have seen a problem in which our variables are not continuous variables, they are binary variables. So, this formulation is a binary integer programming formulation and it is not a linear programming formulation. So, we have used the bin packing to explain a binary integer programming formulation and not a linear programming formulation. Second thing that we learn is that this is a type of a problem where we could start with the solution and if we start with the solution we can actually make the formulation simpler by defining fewer variables and fewer constraints.

So, with this we come to the end of the first module of this course, which is linear programming formulation. In the subsequent module we will look at solving linear programming problems and we will look at that in the next class.