Introduction to Operations Research
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Module - 07
Lecture - 35
Optimality of the Hungarian algorithm

In the last class we saw the dual of the assignment problem. The primal minimizes sigma c I j x I j subject to the condition that one job goes to exactly one person, and one person gets exactly one job. So, in a n by n assignment problem, which means there are n jobs and n people. We have n constraints here, and we have another n constraint here. So, there are two n constraints in the primal.

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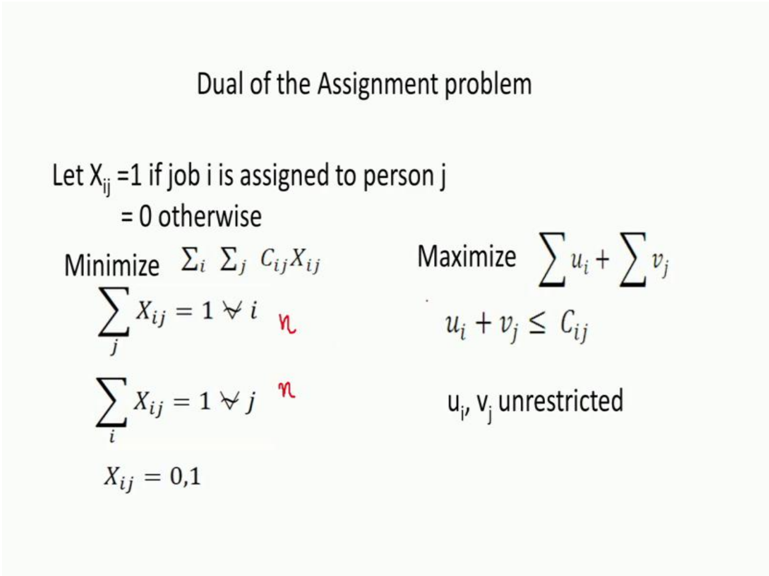

Now, since there are two n constraints in the primal, the dual will have two n variables. So, we will introduce u I variables corresponding to this set of constraints, and v j corresponding to this set of constraints, and therefore, the dual solution, or dual will become maximize sigma u I plus sigma v j, because the right hand side in the primal is one. So, 1 into u I plus 1 into v j sigma u I plus sigma v j u I plus v j less than or equal to c I j dual will have as many constraints as the primal variables. There are n square variables in the primal. There will be n square constraints in the dual, a typical x I j will appear in the i'th constraint here, and the j'th constraint there. So, it will have the form u

I plus v j less than or equal to c I j. So, we also said that we will see how the Hungarian algorithm works, with respect to the dual of the assignment problem.

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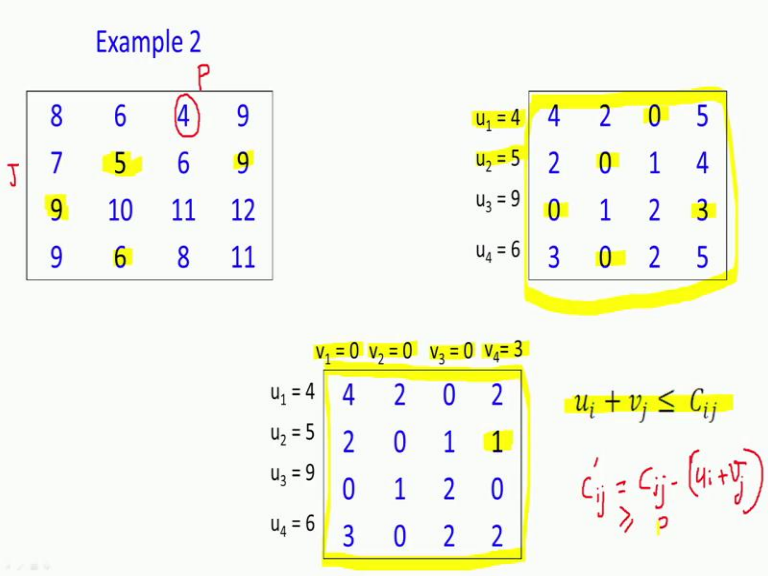

So, let us now go to the examples that we have seen, and explain how the optimality of the Hungarian algorithm happens, or why the algorithm that we have seen gives us the optimum solution. So, let us look at this example which is a 4 by 4 example. So, the four jobs and four persons are here. So, these represent the four jobs, and this represents the four people. So, the first step what we did was, we subtracted the minimum from every row. We identified the row minimum, and subtracted the row minimum from every element of the row. The first row the minimum happened to be here which is four, and then we subtracted this to get 4 2 0 and 5. Now this is the same as, writing u 1 equal to 4 u 1 happens to be the row minimum. So, writing u 1 equal to 4, and subtracting u 1 from every element of the first row. Similarly it is a same as writing u 2 equal to 5. 5 happens to be the minimum, so u 2 equal to 5 and subtracting 5 u 3 equal to 9 and subtracting 9 and u 4 equal to 6 and subtracting 6 from every element.

Now, we reduced it to this matrix, the matrix that is shown here, we reduced it to this matrix, and then we looked at the columns one by one, and we subtracted the column minimum. The first row the minimum happened to be 0; that is the same as writing v 1 is equal to 0, and retaining the same column. For the second column also the minimum happens to be 0, which is either here or here, which is the same as saying b 2 equal to 0.

For the third column, the 0 happens to be the minimum. So, v 3 is 0, and the fourth column the minimum is 3, so we write v 4 is equal to 3, and then we subtract the column minimum from that, to get the new matrix. And we also said that this matrix will have at least one 0 in every row and in every column now, this new matrix that we have got. If we take a typical element which is say here. Now this one has come out of a subtracting 5 at this level, and subtracting three at this level. So, this one is nothing, but 9 minus 5 minus 3. So, this value is actually, this value that we have here is c dash I j if I may call it; c dash I j is equal to the original c I j minus u I plus. So, its c I j minus u I plus v j is what we have here. And if the original matrix, has all cost coefficients greater than or equal to 0; this c I c dash I j which is c I j minus u I plus v j will be greater than or equal to 0. So, this will be greater than or equal to 0. So, c dash I j will be greater than or equal to 0.

Now, when c dash I j is greater than or equal to 0, it implies that u I plus v j is less than or equal to c I j, because c dash I j is greater than equal to 0. Means c I j minus u I plus v j greater than or equal to 0 u I plus v j is less than or equal to c I j. So, the u's and the v's that we have defined through subtracting the row minimum and the column minimum, gives us a a dual feasible solution. So, this set of u's and v's are feasible to the dual. Now we look at this matrix and then we make assignments. Now if we made assignments based on; for example, the first row has only one 0, so we could make an assignment here. The second row had only one 0, so we could make an assignment here. And then this goes, the third row has. The first column has one 0. So, this goes. So, we are able to make three assignments. Now if we were able to make four assignments in this, then we said we would have got the optimum solution. Now the motivation is as follows.

We are going to make assignments only in 0 positions. So, when we make assignments only in 0 positions, corresponding to these positions c I j minus u I plus v j is equal to 0, which means the dual constraint there is satisfied as an equation, which also means that the dual slack is 0. So, if this is to be written as an equation, then we need to add a slack which could be some h I j and that slack will be 0. So, what we are doing is, we are trying to make allocations only in 0 positions which means only in positions where the dual slack is 0, which means we are indirectly applying complimentary slackness. So, we would make an assignment x I j will be basic only when the dual slack is 0. Therefore, x I x x which corresponds to the primal variable into v which corresponds to the dual slack, will be equal to 0, because the dual slack is 0. So, if we are able to make an allocation four. If you are able to get four feasible allocations in this matrix, then we would say that we have got the optimum solution.

For this particular example at this stage we are not able to get the four allocations, we have to do something more, but if we have a matrix where an n by n matrix, where after subtracting the row minimum and the column minimum, we get a reduced matrix, and that reduced matrix means, there are some u i's and v j's that we have defined. The u i's are the row minimum, the v j's are the column minimum, and u I plus v j will be less than or equal to c I j, because by subtracting the row minimum and the column minimum, we make sure that the resultant matrix is greater than or equal to 0, which is here. Therefore, it is dual feasible u I plus v j less than equal to c I j, and u I v j's are unrestricted. So, we do not worry about the sign of this u's and the v's. So, doing the row column subtraction and getting a new matrix, is equivalent of defining a set of u and v such that, the corresponding dual solution is feasible. By making assignments only in 0 positions, complimentary slackness is satisfied, and if we are able to get a feasible solution, then such a feasible solution is optimum. Of course in this example we are not able to get a feasible solution with four allocations. So, we do something more. So, we see what we do there and how we explain this.

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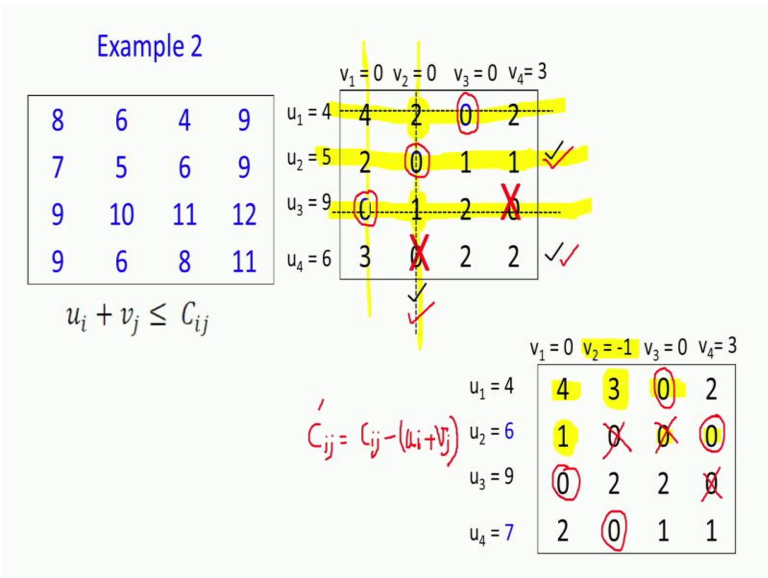

So, we went ahead and we have this, we have this solution. So, we have made assignments in, let us say these three positions we have made assignments. We have made assignments here, we made assignments here, and then we made assignments here. Now these two 0s are not assigned and they are crossed. Now at this point we said that we have not got four assignments, and therefore, we need to try and get the fourth assignment. So, we followed the procedure where we ticked some unassigned rows and row lines. So, we ticked unassigned rows, if there is a 0 in a ticked row then we ticked the corresponding column, we ticked the unassigned rows. If there is a 0 tick that column. And if a ticked column has an assignment, tick the corresponding row. Do this till no more ticking is possible. Draw a line through unticked rows and ticked columns. So, the lines were drawn through unticked rows, these two are the unticked rows, this is a ticked column, we then row these lines. Three lines we drew. And then we also said that these three lines correspond to the fact that we have made three assignments. And we also understood that these three lines are the minimum number of lines required to cover all the 0s. So, we did all that.

Then what we did was, we looked at the other positions where no lines were passing, and then we found out the minimum of them which happened to be one, which is either this one or this one; theta is is equal to 1. And then we said that we will subtract this theta, which is one from every element, which does not have any line passing through. We will add this one to positions where two lines are passing through, and we will retain the rest of them as they are. So, that resulted in this 4 remaining as 4. This two has two lines passing, so 2 became 3. And this two does not have any line passing, so this two became one. These ones became 0 and we got a resultant matrix. At that point we said that we are adding something to some quantities. We are subtracting from some positions, and we do not do anything to some other positions. Another way of looking at it is, if we take a typical row. Here the row is either ticked or has a line passing through. So, if the row is ticked, and does not have a line passing through the entire row then, it will either have elements where only one line is passing, or elements where no line is passing. So, if we take this typical row, then we would have subtracted from some positions, and kept some positions the same, these 0 positions the same. If we look at this row, which for example, has a line passing through. Then we would either keep the some positions same and add. So, that is what happens in the rows, if we take a row, we either do this or that. We either keep certain things fixed and subtract, or keep certain things fixed and add. So, that is the change that we make in the new column.

Similarly, if we take a column, if we take this column where a line is actually passing through, we either keep things as they are or we add. If you take this column where line is not passing through, then we either subtract or keep things as they are. So, this process of adding the theta to the intersection elements subtracting the theta from elements where no lines is passing through, and keeping the rest of them as they are, is like redefining the u's and the v's . Now the original u's were 4 5 9 and 6. Now the new u's are 4 6 9 and 7. How did we get this? These are the positions where there are ticks, which means these are positions or rows where there is going to be no line, because we draw a line only through unticked rows. So, it is a ticked row. So, there is going to be no line. So, if there is no line passing through a row, then add theta to that u. So, 5 becomes 6, 6 becomes 7. If there is a line passing through that row, then keep the u as it is. So, u 1 becomes 4, u 2 becomes 6, u 3 becomes 9, u 3 has a line. So, the u remains, u 4 does not have a line, so u 4 is equal to u 4 plus theta, so 6 will become 7. Similarly if you take the four columns if the column does not have a line passing through keep the v as it is. So, v 1 is 0, v 3 is 0, v 4 is 3. If a column has a line passing through then subtract theta from that v. So, v 2 becomes minus 1.

So, now we have a new set of u's and v's which correspond to the new table. Now when we have this new set of u's and v's, the way we have defined it, we now observe that the new set of u's and v's are such that u I plus v j is less than or equal to c I j. We now realized that this set of values that we have now. If we call this as c dash I j. If we call this as c dash I j, then this c dash I j is equal to c I j minus u I plus v j. Let me explain this for example, if we take this element the 2 has become 3 here. So, let us see what happens. Now, this 3 is c I j minus u I plus v j; u I plus v j is 4 minus 1 3 c I j is 6 6 minus 3 is 3. Suppose I take this four, where no change has happened u I plus v j is 4 8 minus 4 is 4. If we look at this one, where 2 has become 1 u I plus v j is 6 plus 0 6 7 minus 6 is 1. So, the transformation from this matrix to this matrix what actually happens is, we look at this matrix and after drawing the lines, wherever an element has two lines passing through we add the theta. Wherever one line passes through we keep it as it is. No line passes through we subtract theta. So, it looks as if we are doing something, in some places we are adding, some places subtracting and so on, but there is a order and there is a reason for doing this.

Now, this whole thing of adding theta to the intersection element, subtracting theta from elements that have no lines, and keeping the one line elements. Line elements that have one line passing through them, the same, is equivalent to redefining the u's and v's such that u I is u I; the same u I remains if the row has a line, and u I is equal to u I plus theta, if the row does not have a line. It is also equal to v j is equal to v j when there is no line, which means the v's are the same. And v j is equal to v j minus theta when there is a line passing through a column. And by redefining the u's and v's we are now creating another c I j minus u I plus v j matrix. And we also make sure that this matrix is greater than or equal to 0. Every element in this matrix is greater than or equal to 0. Which means this matrix also satisfies u I plus v j less than or equal to c I j. Therefore, the new set of dual variables, which are got from the original set of dual variables by theta adding or subtracting or retaining the theta, is also a feasible to the dual. And it is also interesting that u i's and v j's are unrestricted in signs. So, we are not really worried about the fact that the v 2 has taken a minus 1, because by definition u I and v j are unrestricted in sign.

Therefore the new matrix that we have obtained, is also dual feasible. And the new matrix that we have obtained now has a corresponding u and v such that u I plus v j is less than or equal to c I j. Now in the new matrix we would only make assignments in 0 positions. For example, we look at the first row, and we say that we can make an assignment here, which means this 0 will go. We temporarily leave out the second and the third rows we make an assignment here, this goes. And then we make an assignment here, this goes. And we can make an assignment here to get four assignments, and therefore, we have got the optimum solution. Why, because we make assignments in 0 positions. So, once again in a 0 position c I j minus u I plus v j is equal to 0. Therefore, u I plus v j is equal to c I j; the constraint is satisfied as an equation, the corresponding dual slack is 0. So, the a primal variable can be basic only when the dual slack is 0. We make assignments only in 0 positions. Therefore, we satisfy complimentary slackness conditions. Therefore, the moment we get a feasible solution with four assignments it is indeed optimum, but we also do something interesting between this table and the other table. Now by looking at this theta; theta is equal to 1 here, theta is equal to 1 here. And look at this position we actually in this position, we increase the number. Even if we had a 0, and if two lines were passing through, we would increase it.

Now when we apply this procedure and we see, if we actually had a 0 here, instead of this one, allocating into that 0 is going to take away the chance of doing this and doing this. A 0 that now has two lines passing through, is actually an unwanted 0, and that is not going to help; therefore, we add. But more than that what we ensure is, from here there are two places where we have this one, and these two have become 0 in the next iteration. So, we create at least one new 0 in the next table. We create at least a 0 in a new position in the next table which is a case where, or which provides us with an opportunity to change the solution, and perhaps get an extra allocation. So, this idea of having a dual feasible solution, ensuring that the dual solution is feasible, ensuring complimentary slackness, and getting an optimum solution, when the primal becomes feasible, generally comes in category of what are called dual algorithms. Now here by changing the dual, from one set of dual variables to another, and relating them through a parameter called theta and getting a new dual solution. And also making sure that the new dual solution gives us an opportunity to have a different allocation in the primal, is what is called a primal dual algorithm, and this Hungarian algorithm that we have seen, is an example of a primal dual algorithm, but the basic principle remains the same.

The Hungarian algorithm works on the principle that, by doing this row minimum and column minimum subtraction, we are defining dual variables. And by ensuring that after the row column subtraction, the matrix is greater than or equal to 0, we are ensuring feasibility to the dual. By allocating in 0 positions, we are maintaining complimentary slackness, and when therefore, the primal is feasible it is optimum. Now when the given matrix is unable to give us a feasible solution, we do this lines, we draw the lines, and then we subtract theta from some positions, we add theta to some other positions, and keep other positions the same, and that is equivalent to defining a new set of dual variables consistently; such that the new set of dual variables is also a feasible, because the new matrix is also greater than or equal to 0. But the new set of dual variables gives us an opportunity to create new assignments, and that is how the Hungarian algorithm actually works. In the next module, we will introduce how to solve linear programming problems up to transportation and assignment, using some solver. So, far we have solved all these problems by hand or by algorithms, and we have explained the algorithms. Now we will also explain how we can use solver to try and solve these problems. So, the next set of module, and the next set of classes will address this issue.