Introduction to Operations Research
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Module - 07
Lecture - 34
Unequal number of rows and columns; Dual of the assignment problem

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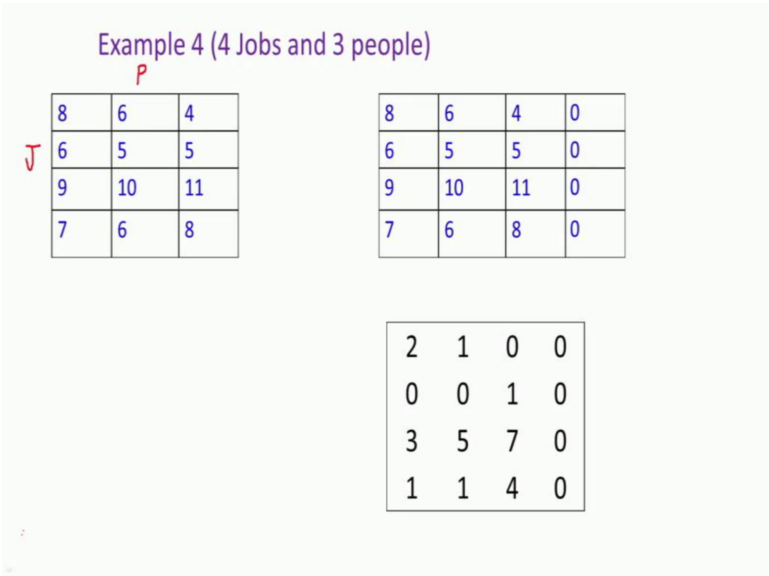

In this class, we will look at an example, where we have an unequal number of jobs and people. So, let us look at this example, where we used the rows to represent the jobs, and we used the columns to represent the people. So, here we have a situation where, there are four jobs, but there are three people. Now how do we solve an assignment problem? Can we solve an assignment problem? Now in this case we will relax it a little bit and say that, each job will go to only one person, becomes difficult, because there are an unequal number of jobs and people, but each person will get only one job is possible, because there are three people, and we now choose the three out of the four jobs that have to be given to the three people. So, which means one job is not going to go to anybody. Now which job is not going to go to anybody, which three are going; such that the cost is minimized, is the new assignment problem, in a situation where there are unequal number of jobs and people.

As I already mentioned it need not be jobs and people all the time; it can be jobs and machines, it can be students and projects and so on. Can be companies and projects, companies and tasks? It could be anything, but there unequal number of these two entities. First thing we do is to make an equal number of entities, by creating a fourth person; an imaginary fourth person, who is going to charge 0 to do each of these jobs. So, this four by three problem; four job three person problem, has now been made into a four job four person problem, by adding what is called a dummy column, what is called a dummy column. And the cost in the dummy column is 0, which means the fourth person, who is an imaginary person, is going to charge 0 to do every job. Now we subtract the row minimum and the column minimum.

So, we subtract the row minimum; the row minimum for all the rows the row minimum is 0, because every row now already has a 0. Therefore, we do only column minimum. To do this subtraction the first column, the column minimum is, first column, the colour minimum is here, so it becomes 2 0 3 1. Second column the column minimum is here 1 0 5 1. Third column the column minimum is here, 0 1 7 and 4. Of course the fourth column has all 0s. So, this is the reduced matrix after the column subtraction. We do not do row subtraction, because the row minimum is already 0 for every row. Now we start making an assignment with this 4 by 4 matrix.

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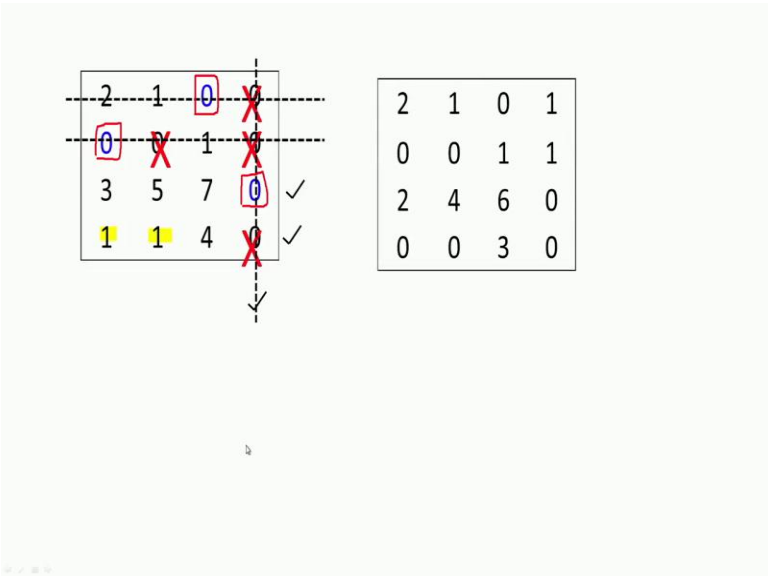

So, this is the matrix. So, the first row has two 0's, so we do not make an assignment. The second row has three 0s, so we do not make an assignment. The third row has only one 0, we make the assignment. And then we cross this 0, this goes, and this goes, because an assignment has already been made to the fourth column. So, the other 0s become. So, all the other 0s go. Now we look at the fourth row, there is no assignable 0. Now we look at the first column. So, we had already made our first assignment here. The first column has only one 0 assignable 0. So, we make an assignment here, and then this 0 will go. So, this 0 will go. Now, second column does not have an assignable 0. The third column has only one 0, so we will make an assignment here. To do that we again put all these assignments in the boxes. This was our first assignment, this was our second assignment, and this is our third assignment. Now all the 0s are either assigned or crossed, but then we have only three assignments, as against the required four, so we do the ticking and line drawing procedure. So, we do that, tick an unassigned row.

So, the fourth row does not have an assignment; tick an unassigned row. If there is a 0 tick that column. So, tick the fourth column. If there is an assignment, tick that row, tick the third row more ticking is possible. Draw a line through unticked rows and ticked columns. So, these are the two unticked rows, and ticked column. There are three lines indicating three assignments, and all the 0s are covered by the lines. So, the minimum of the uncovered element, is one which is happens to be here, which happens to be here. So, theta is one, and now redo this matrix by adding theta to the intersection elements; these two, subtracting theta from the uncovered elements, and keeping the rest of them as it is. So, we do that to get the new matrix. Now, this 2 1 0 will remain. This 0 will become one, theta is added 0 0 1 remains, this 0 will become one theta is added 2 4 6 0 2 4 6 0 0 0 3 0. So, this is the reduced matrix. Now start making assignments in this reduced matrix.

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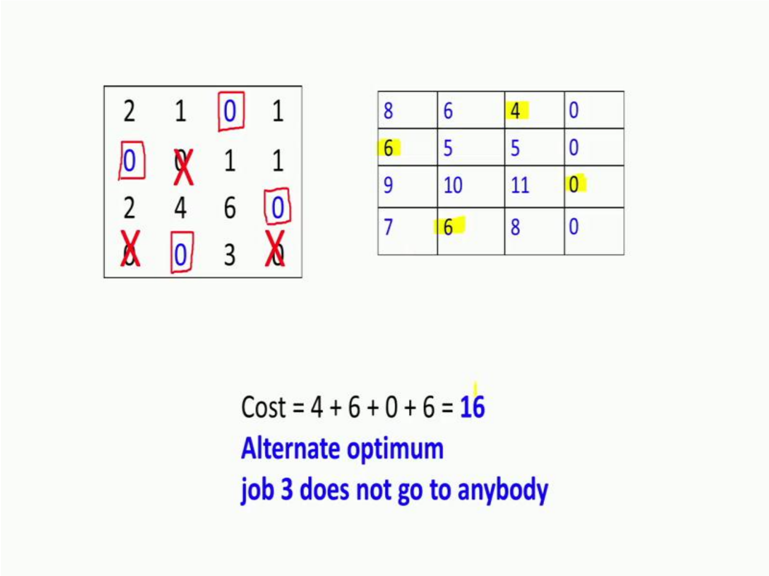

So, once again, the first row has only one 0, so make an assignment now. The second row has two 0s, so leave it temporarily. The third row has only one 0. Make an assignment, this 0 goes. The fourth row has two 0s. Do not make an assignment. The first column again has two 0s, again has two 0s. So, do not make an assignment. Second column has two 0s. Do not make an assignment no more assignments are possible. We reach the tie situation, or we reach the alternate optima situation. So, this tie has to be broken arbitrarily, and we do that. So, again to mark the assignments; this was our first assignment, this was our second assignment. Let us assume we break this arbitrarily to do this. So, this and these two go and this becomes our fourth assignment. Now the total cost will be corresponding to these, to this assignment the cost is four. Corresponding to this the cost is six, corresponding to the third the cost is 0, corresponding to the fourth the cost is six. So, the cost is 4 plus 6 10 plus 0 plus 6 is 16. If we had broken it arbitrarily instead of getting 6 plus 6, we would have got 7 plus 5, which would have given us the same twelve and the same cost, but an alternate optimum solution.

Now, from this we also understand that; since the… Now this was our dummy, the imaginary, fourth one was the imaginary one. So, the dummy is now assigned to job number three. So, job number three goes to the imaginary position; therefore, job number three does not go to anybody. The other three jobs go to the three persons, this, this, and this, or it could be this this and this. So, when you have an unequal number of jobs and people. If you have more jobs than people, then some jobs at the end will not be assigned to anybody. If you have more people than jobs, then some people will not get jobs. Now, when you have an unequal number of jobs and people, or any two entities, you first have to make them equal by adding corresponding dummy's. A four row three person, or four job three person problem will now have a fourth person, which is a column. If I had three jobs and four persons, then they there will be a dummy row. The dummy row or a dummy column, sometimes more than one, will have cost equal to 0, and then solve the assignment problem, till you get the optimum solution. And finally, realize whether a job is not going to anybody, or a person is not getting a job, depending on which one was more than the other. So, this is how you solve an assignment problem when you have an unequal number of jobs and people, or an unequal number of both these tasks.

Now, we have seen the assignment problem solution by actually considering four examples; the first was the state forward example, where the row minimum column minimum subtraction gave us a feasible solution, and which was optimum. Second example the first row minimum column minimum did not give us a feasible solution. So, we learnt the ticking and line drawing procedure through that. The third example we got into a situation where, we were not able to make assignments 0s were available. So, we had to make arbitrary assignments, which lead to alternate optimum solutions. The fourth example is where we had an unequal number of jobs and people and we solved it. So, in all this we have used algorithms, which involve largely addition and subtraction, and we have used algorithms which primarily work on two principles.

If I have a matrix where, all the coefficients are non-negative, if I make assignments only in 0 positions, if I get a feasible solution, then it is optimum. The way I get the 0 positions, or based on subtracting the row minimum and column minimum, under the premise that the solution will not change. The only place where it looks as if we deviated was, when we drew those lines, and then we added something to some places, we subtracted something, and we kept something the same, and yet said the solution to the new matrix, is the same as the solution to the old matrix, and we will have to see how that happens. We also have to see that are we going to use some ideas of linear programming. Are we going to use ideas from dual, to solve the assignment problem? Though at the moment what we seem to have done, does not seem to relate directly to linear program. Now we will answer these two questions now, by looking at going back and revisiting the formulation of the assignment problem, and then by writing its dual. So, let us do this.

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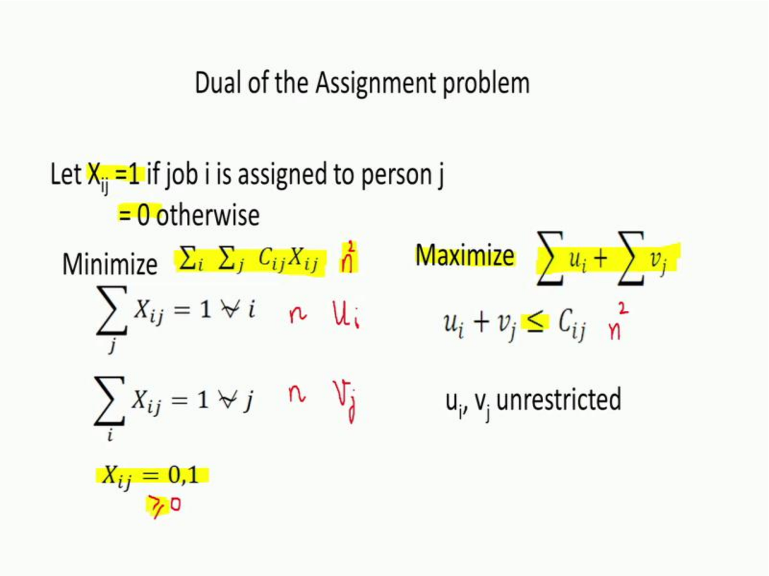

So, now let us write the dual of the assignment problem. So, first we write the primal which is, let x I j be equal to 1, if job I is assigned to person j and its equal to 0 otherwise. So, we said the decision variable to the assignment problem at this point, is a binary decision, its equal to 1, or it is equal to 0. It is not a continuous variable. The objective is to minimize the total cost of assignment, so given by summed over I, summed over j, summed over the jobs, summed over the persons c I j x I j. Now, the first constraint is for every job. So, for every job x I j equal to 1, for every person j summed over I, so every person j gets only one job out of the n jobs. The second set of constraint says, for every job it will go to only one person. So, the two important sets of constraints, each have n constraints if the assignment is a n by n n problem; n jobs and n people. There are n square decision variables, because x I j I equal to 1 to n j equal to 1 to n. There are n square decision variables, there are n plus n 2 n constraints. Out of these square or sixteen variables, only four variables will be the solution. Only four variables will take one. And these four variables will be such that every row has one variable and every column has one variable. There are n factorial possible solutions, and we want to find out the best out of the n factorial possible solutions.

Now, we said that this is a binary problem. So, to relate it to linear programming, we will now change it to a continuous variable. We will assume that x I j is greater than or equal to 0 because when we actually solved the assignment problem, by the method that we saw. The method that we saw is called the Hungarian algorithm. So, when we use the Hungarian algorithm, we realize that we are getting the binary assignments. So, we will make an assumption like in the transportation, that solving the assignment problem as a l p would give us binary solutions, would give us one or 0. I have not highlighted the unimodularity property, though I mentioned it when we looked at transportation, the same thing is true with the assignment. So, if we solve it as a l p, you will still get binary values. You will get a 0 or a one. So, we treat this variable now from a binary variable to a continuous variable.

Now, the moment we treat it as a continuous variable, it becomes a linear programming problem, objective function is linear, constraints are linear, variables are continuous, and there is an explicit non negativity restriction on the variables. So, we can now write the dual. Now when we write the dual, we realize that there are n square variables here, n square variables here. There are n constraints here, there are another n constraints here. Now these n constraints we define n dual variables, which are called u I, and then we also define another n variables which are v j. So, there are n u i's u 1 to u n, there are v j's v 1 to v n. There will be as many dual variables as the primal constraints. So, there are n u i's and then there are n v j's. Now, the objective function of the dual, it is a maximization problem, because the primal is the minimization problem. The dual is a maximization problem. The objective function is the right hand side of the primal so, 1 into u I plus 1 into v j, so maximize sigma u I plus sigma v j. Now there will be as many dual constraints as the number of primal variables. There are n square primal variables, there will be n square dual constraints. So, in this example, there are four by four if we take, there will be sixteen dual constraints. Now they will be of the form u I plus v j less than or equal to c I j; that is because, if I take a typical x I j it will appear here in the i'th constraint, and here in the j'th constraint. So, it will take the form u I plus v j. The objective function is c I j. So, u I plus v j less than or equal to c I j, maximization problem less than or equal to, because minimization problem greater than or equal to variable.

So, greater than or equal to variable will result in the corresponding less than or equal to constraint in the dual, which is the maximization problem. So, u I plus v j less than or equal to c I j and there are n square or sixteen such constraints. Once again u i's and v j's are unrestricted, because the primal constraints are all equations. We have already seen in our duality that a primal constraint is an equation, the corresponding dual variable will be unrestricted. What is interesting at the moment is, the dual of the assignment problem, looks very similar to the dual of the transportation problem, except that in the transportation problem we had sigma a I u I plus b j v j. But u I plus v j less than or equal to c I j was is common to both the transportation and the assignment. It is also not surprising, because the assignment problem is a special case of a transportation problem. We have already seen that the problem is a transportation problem with an equal number of supply points and demand points, and each supply and each demand having one unit.

The most important thing is this u I plus v j less than or equal to c I j, and we will use this, and we will show how we have actually used this. So, we have now written the dual of the assignment problem. And we will use this dual of the assignment problem to explain why and how the Hungarian algorithm is actually optimum. So, whatever we did in terms of row minimum subtraction, column minimum subtraction, drawing these lines, putting these ticks across the rows and columns. They are all related to the u's and the v's. And how they are related to the u's, and the v's how and why the Hungarian algorithm is indeed optimum, we will see this in the next class.