Introduction to Operations Research
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Module - 07
Assignment Problem
Lecture - 03
Hungarian Algorithm; Alternate Optimum

In the last class, we were solving a second example and this 4 by 4 matrix is shown here. We subtracted the row minimum to get the first reduced matrix, and then we subtracted the column minimum to get the second reduced matrix. Now, the second reduced matrix will have at least one 0 in every row and every column and we make assignments in it.

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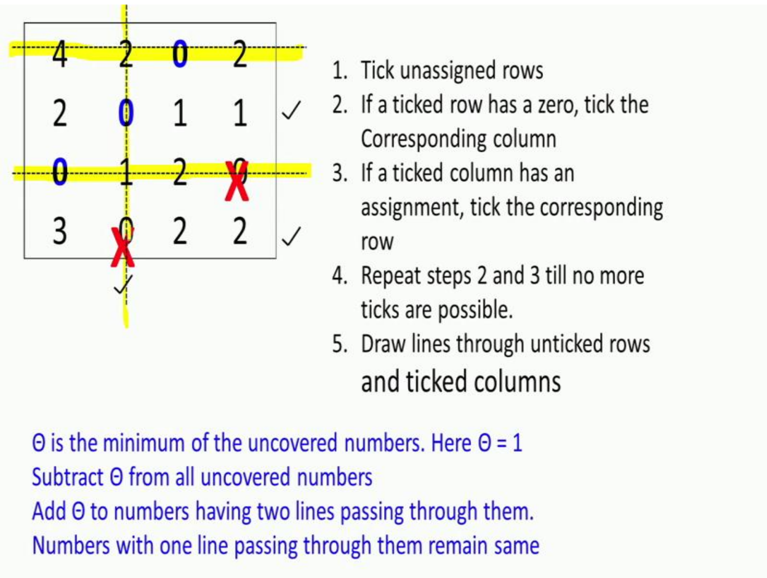

So, we started with the first row, there is only one 0. So, an assignment has been made, and then we went to the 2nd row, which has only one 0. So, an assignment is made and this is crossed. The 3rd row has two 0's, so we do not make an assignment to begin with. The 4th row does not have an assignable 0, so we went to the 1st column, and then we made an assignment there. Now, that removes or puts an X in this position, which we got here.

Now, at this point, we said that, we cannot make any more assignments, because all the 0's have either been assigned or have been crossed, which means they are not fit for assignment, but there are only 3 assignments has against the expected 4. So, we do something now. So, we tick an unassigned row, this was the first tick and if there is a 0 in a row that is already ticked, then tick the corresponding column, so there is a 0 here. So, we tick the corresponding column and we get this.

Now, if there is a column that is already ticked and it has an assignment, then tick the corresponding row. So, there is an assignment here, so we will make a tick here, so we get this tick. Now, we have the rule is, if there is a row that is already ticked, if it has a 0, tick the column. So, this row has 0, but the column is already ticked, this row has a tick and a 0, the column is already ticked. If a column is already ticked and it has an assignment, then tick the corresponding row, it is already done. So, no more ticking is possible.

Now, draw lines through unticked rows and ticked columns, which is what we did here, draw a lines through unticked rows, this row and ticked columns, this two unticked rows and here is a ticked column. So, draw lines through unticked rows and ticked columns. Now, two interesting things happened, we drew three lines and there were three assignments. So, there is a relationship between the number of lines that we draw and the assignments.

And not only that, all the 0's have been covered by either one line or two lines, in this case, all the 0's are covered by one line. We can also show that for this matrix, we need minimum of three lines to cover all the 0's. So, this ticking and line drawing is a very algorithmic way of trying to cover all the 0's with minimum number of lines. So, if there are three 0's, then we will be able to do it with three lines or alternately if we get three lines, then a maximum of three assignments are possible. So, we have done this.

Now, we look at this, now all the 0's are covered either by one line or two lines. So, every position that does not have any line passing through will now have a strictly positive number. For example, 2, 1, 1, 2, 2, 3; that is because the 0's already have one line or two lines passing through them. Now, take the minimum of the numbers, where no line is passing through, which means take the minimum of this 2, 1, 1, 3, 2, 2. So, the minimum is 1, it happens in two places and the minimum has to be a positive number, because the 0's are all covered with one line or two lines. So, you will have a positive number, so that minimum is called theta, so theta is equal to 1.

Now, what you do is, go to this matrix, wherever two lines are passing, example this position add the theta. So, 2 becomes 3, add the theta, 1 becomes 2, where there is only 1 line passing, do not do anything, 4 remains as 4, 0 remains as 0, 2 remains as 2 and so on. Wherever, no line is passing, subtract the theta, so this 2 will become 1, this 1 will become 0, this 1 will become 0, this 3 will become 2, this 2 will become 1 and this 2 will become 1. So, make this modification and create a new matrix.

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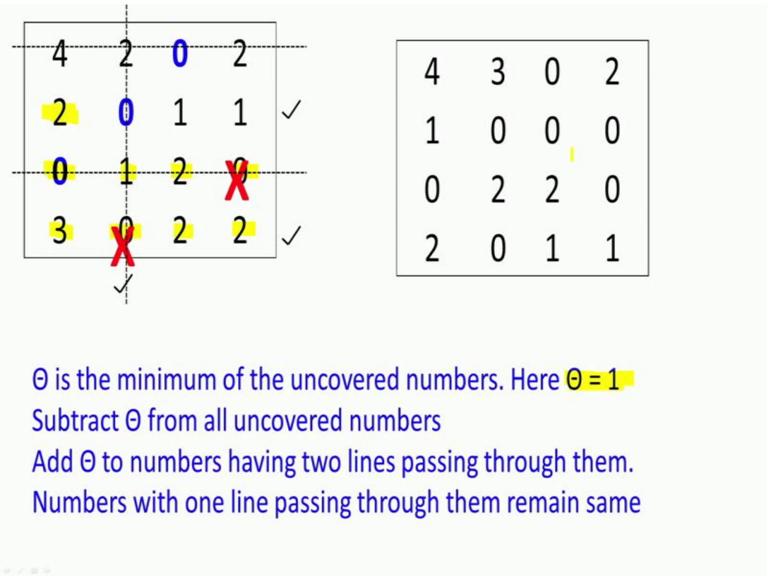

So, I am just showing that modification, again I am showing the ticks and all that the minimum is 1. So, 4 has only one line passing, 4 remains as 4, 2 has 2 lines passing, 2 becomes 2 plus 1 3, because theta is equal to 1, so 2 becomes 3, 0 remains as 0, 2 remains as 2. This 2 becomes 1, because there is no line passing through, 0 remains as 0, 1 becomes 0, 1 becomes 0. This third 0 remains as 0, 1 becomes 2, 2 remains as 2, 0 remains as 0, 3 becomes 2, 0 remains as 0, 2 becomes 1 and 2 becomes 1. Now, this is a new matrix that we have got from the previous matrix.

Now, I am going to say that the solution to this matrix and the solution to this matrix are the same. At the moment, it becomes a little difficult to intuitively get that, when we said the solution to this matrix and the original matrix are the same. We made an assumption that if we add or subtract a constant, particularly subtract or sometimes add a constant, the solution does not change. Here, we are not consistently adding or subtracting the same constant, some places we are adding in a row, some places we are keeping it as it is.

If we take some other row, some places we are subtracting and some places we are keeping it as it is. If you take a column I either subtract or I keep it the same or I add and I keep it the same. So, within a row or within a column in some places, I am adding and some places I am subtracting. But, nevertheless I am going to make that assumption and later show why that assumption is true that, whether I solve this, the new matrix, I will get the same solution. So, I start making assignments with the new matrix.

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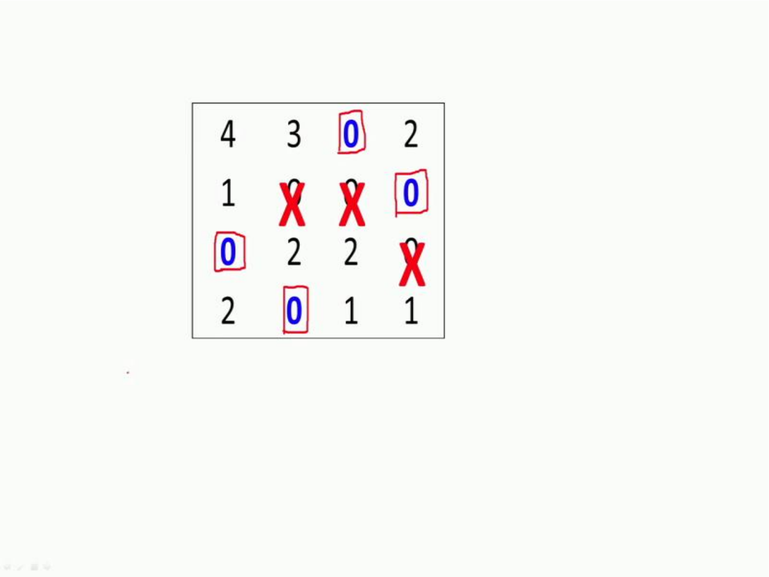

So, I look at the first row, the first row has only one 0, so I make an assignment and when I make an assignment, this 0 will go. So, that is shown by this X coming. Now, the 2nd row has 2 assignments, so I do not make an assignment right now. The 3rd row again has 2 assignments, so I do not make an assignment right now. The 4th row has only 1 assignment, so I make an assignment here, I am again going to show both the assignments. This was our first assignment and this is our second assignment and when we make this assignment, this will also go, so that is shown by another X coming here.

Now, I look at column wise, the 1st column has one 0, so I can make an assignment. So, again I am going to put those assignments in the boxes, want to put the assignment, this was our first assignment, this is our second assignment. Now, the column has only one 0, so make an assignment, and then cross this which is shown by this X coming. Now, the second column is already assigned, the 3rd column is already assigned, the 4th column I can make an assignment here, which is shown here.

So, right now it shows only the blue color. So, I am just putting it into the box to show this was our first assignment, this is our second assignment, this is our third assignment and this is our 4th assignment, because this column or this row has only one assignable 0. So, now, we have got 4 assignments, and therefore we have got a solution.

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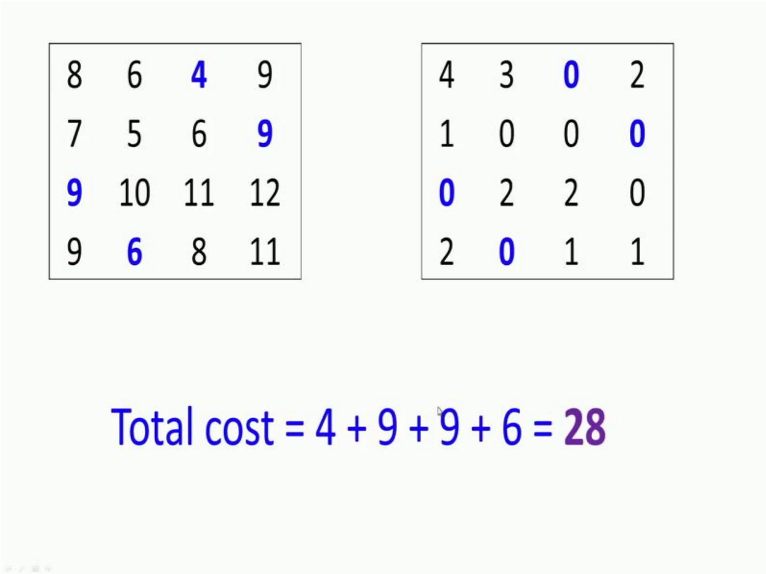

So, these were the 4 assignments which are shown in blue color on this side and the corresponding, these are just to highlight, these are the assignments 1, 2, 3 and 4. The corresponding costs are 4 plus 9, 13 plus 9, 22 plus 6, 28. So, we have now solved this assignment problem to get the best solution. So, now, what is the algorithm? The algorithm is start with the matrix, first the matrix should have all costs greater than or equal to 0, subtract row minimum from every row, subtract column minimum from every column, make assignments.

When you make assignments, if a row has only 1 assignable 0 or a column has only 1 assignable 0, you will make the assignment. If it has more than 1 assignable 0, you will not make the assignment. You will continue to make assignments, till all the 0's are either assigned or crossed. If you have n assignments in this case 4, you have reached the solution. If you do not have, then there is at least 1 row that is not assigned.

So, proceed to do the ticking, tick an unassigned row, if there is a 0, tick that column. If a ticked column has an assignment, go back and tick that row and keep repeating it. Draw lines through unticked rows and ticked columns, all the 0's will be covered by lines. Now, look at those numbers, which are not covered by a line, those numbers will all be positive numbers. Take the minimum of them, add this minimum to positions, which has two lines, subtract this minimum from positions that have no line.

And keep the rest of them the same, which means, numbers that have one line passing through them, they remain the same. Get a reduced matrix; start making assignments, till you get 4 assignments or n assignments. If required go back again to do the ticking and line drawing procedure. So, here in one iteration, one ticking and line drawing, we got the solution and there will be situations, where that has to be done more than once. So, we have now solved two problems to understand the algorithm almost fully.

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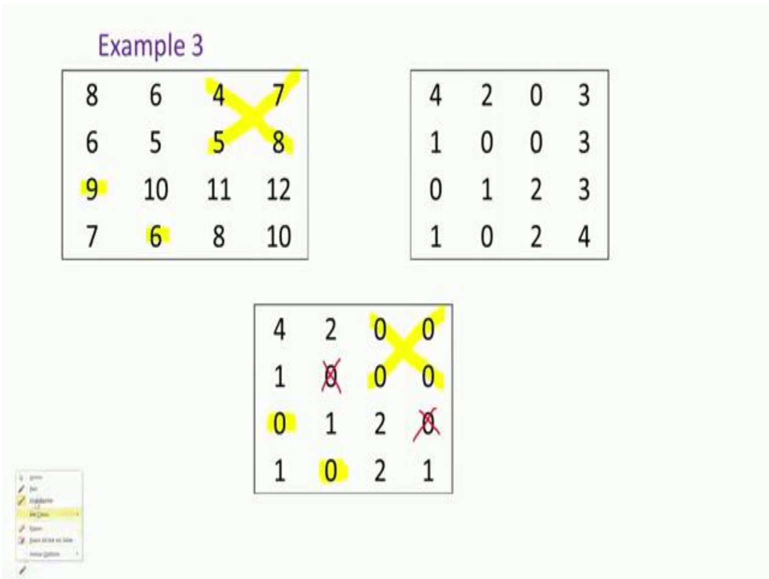

Now, let us look at a 3rd example, where we try and understand a few things. Now, we take a 3rd example, now this third example again has a 4 by 4 assignment problem. We do the row minimum subtraction of 4, 5, 9 and 6. We also find that the first 3 columns have 0's, but the 4th column does not have a 0. So, subtract the column minimum from the 4 to get this reduced matrix. So, this is our reduced matrix. So, start making assignments in the reduced matrix.

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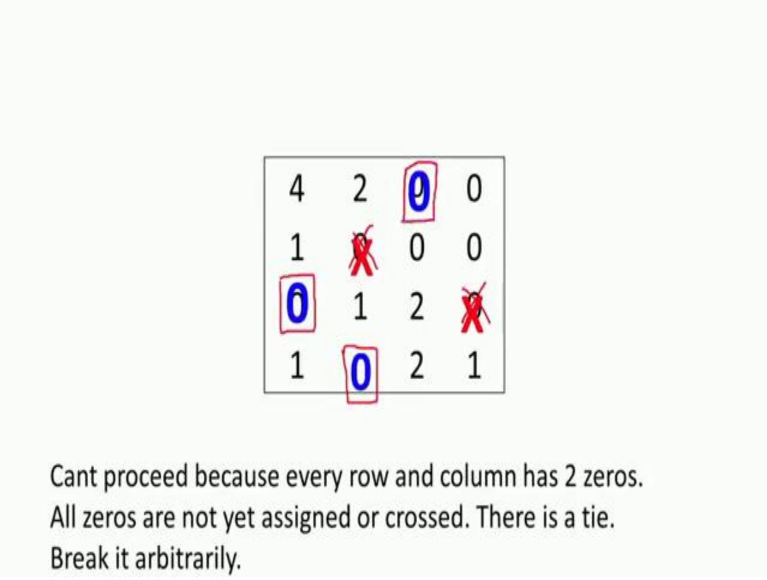

Now, the first row has two 0's,, therefore do not make an assignment. The 2nd row has three 0's, so do not make an assignment. The 3rd row has two 0's, so do not make an assignment, but the 4th row has only one 0, so make an assignment here. So, this is our assignment and this goes, because we have made the assignment, this will go. Now, this is goes, now look at the 1st column, the 1st column has only one assignable 0, so make the assignment.

So, the 1st column has an assignment now, I will also show it using a box and this will go this also goes. Now, go back and see, second column is already assigned, 3rd column has two 0's, 4th column has two 0's, so do not assign. Go back, first row has two 0's, 2nd row has two 0's, so do not. Now, we have a situation, where all the 0's are not boxed or crossed, boxed means assigned, crossed means not fit for assignment. But, now the 0's are hanging, and therefore we cannot start our ticking and line drawing procedure.

So, this is the third thing that can happen. So, if we have this thing that is happening, then we have to resolve it in some way and we make an arbitrary assignment. So, we make an arbitrary assignment, we cannot proceed, because every row and every column has two 0's all the 0's are not yet assigned or crossed, there is a tie now. So, break it arbitrarily and make an arbitrary assignment. Now, we say that we are going to make an assignment here. So, this is an arbitrary assignment, we could have assigned in any one of the 4.

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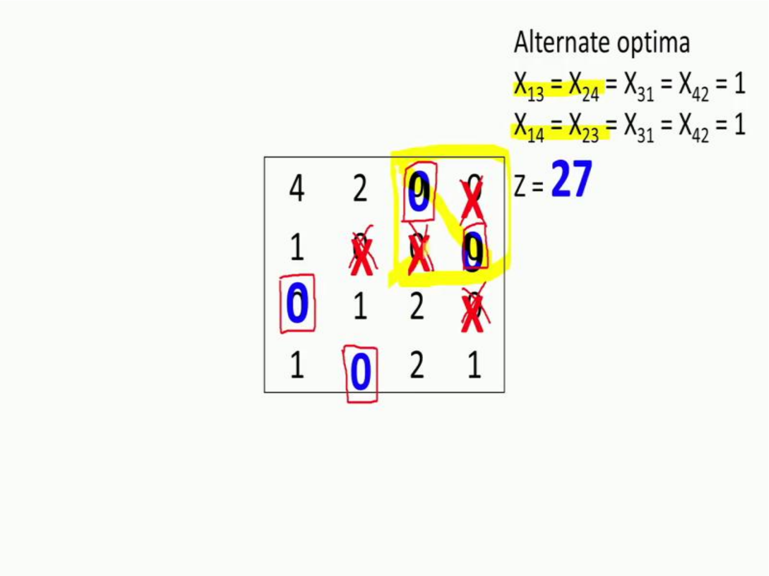

So, we make this assignment, this automatically goes, this will go and this will go. By making that assignment, this will also go and this will also go. So, now, the 2nd row or the 4th column has only one assignable 0 and make the assignment there, has only one assignable 0 and make the assignment there.

So, what now we did is, when we reached this point, when we had two 0's, we could have broken that tie arbitrarily by choosing any one of the 4, we chose this, we automatically got this. If we had chosen this or this, we would have got the other. So, there are two possibilities and any arbitrary way of breaking it would have given one out of the 2. So, we have alternate optimum.

So, one possibility is 1, 3 and 2, 4 are in the solution, one part is X 1 3, X 2 4 is in the solution, the other is X 1 4 and X 2 3 are in the solution. For example, the first solution is about having these 2, 1, 3 and 2, 4 are in the solution, second solution is 1, 4 and 2, 3 are in the solution. So, there are alternate optima and both of them will give a value of 27. For example, if we go back to the starting solution, so if we go back to this ((Refer Time: 17:38)), we realize that the actual solution happened to be.

When we make these assignments, we realize that, this was our first assignment. This would go this was our second assignment, this would go, and then we had the tie situation. So, we could either pick this combination, the other two would go or we could pick this combination. So, our cost would be this 9 and this 6, corresponding to these two positions and it is either 4 plus 8 or it is 5 plus 7. So, 4 plus 8, 12, 12 plus 9, 21 plus 6, 27 and 4 plus 8 is equal to 5 plus 7. So, we would get alternate optimum and all of them having 27 as the value.

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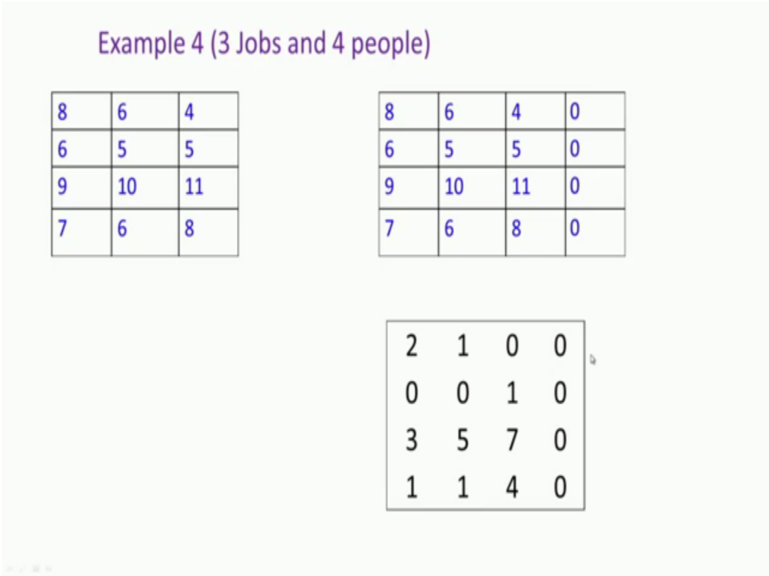

So, all of them would give us both the alternate optima would give us solutions with 27 as the value. There are still a couple of more aspects to the assignment problem. Fact till now we looked at assignment problems that have an equal number of people and jobs. We looked at a 4 by 4 in all our examples till now. Now, can there be situations, where there are only three jobs, but there are four people. Can there be a situation, where there are four jobs and there are only three people. Now, which means can we have a situation, where we have an unequal number of the two entities. Can we still solve it as an assignment problem is the next thing that we will be looking at? We will look at that in the next class.