Introduction to Operations Research
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Module - 07
Assignment Problem
Lecture - 02
Solving the Assignment Problem

In the last class, we introduced the assignment problem and we started solving a 4 by 4 assignment problem, which is shown on the left hand side.

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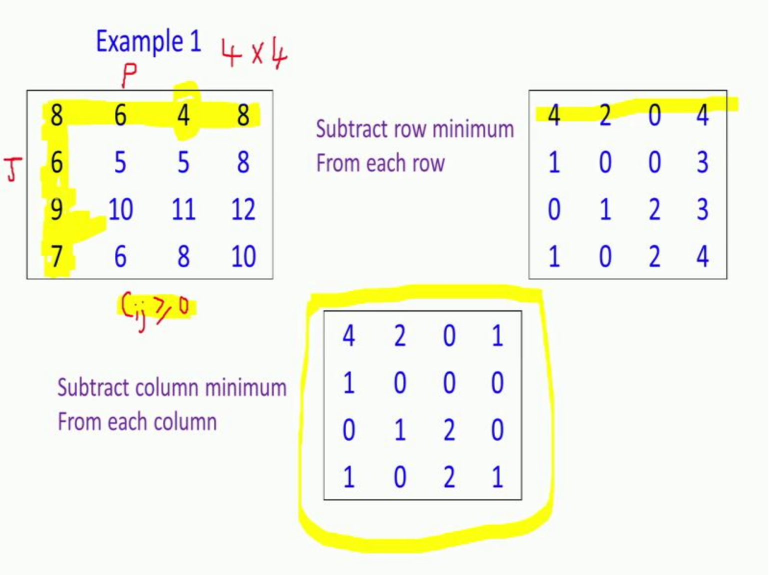

We said that the solution will depend on two very simple ideas. The first one being, if we make assignments in 0 positions and if we are able to get a feasible solution, which means every row has an assignment and every column has an assignment. Then, such a solution will have 0 cost, and therefore will be optimum, so long as all costs are greater than or equal to 0.

We also observed that there are no 0 cost in this matrix. So, we said we will now reduce this matrix to another matrix and we did that by subtracting the row minimum from every row and column minimum from every column and got this resultant matrix. Now, this resultant matrix which we have here, now this resultant matrix that we have here, now has a 0 in every row and has a 0 in every column.

Therefore, this is a matrix using which we can make the assignments and we try to make the assignments in 0 positions and see, whether we can get a feasible solution, which means, we will now make assignments only in the 0 positions. And then see whether we can get 4 assignments, such that every row has 1 assignment and every column has 1 assignment, now let us do that.

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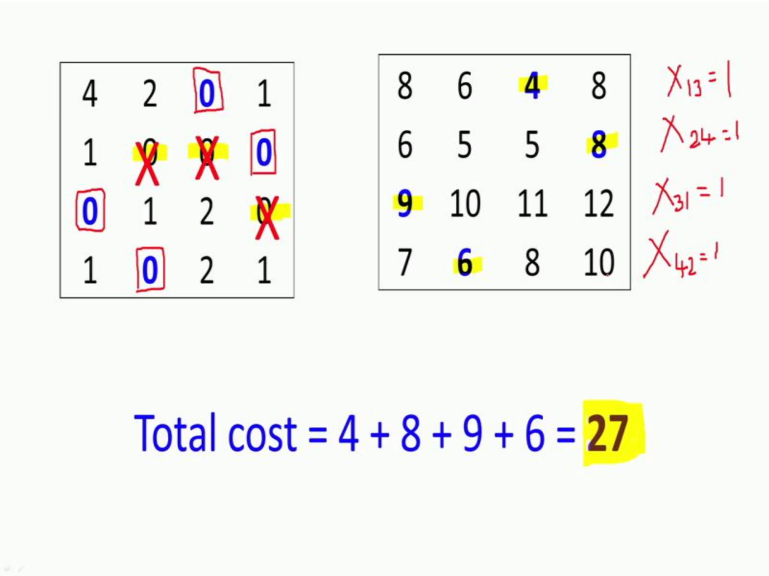

Now, let us take this matrix and start making the assignments. So, the same matrix is shown here. The 1st row, we observe that 4, 2, 0 and 1; the 1st row has only one 0, and therefore we can make an assignment here, which is shown in blue color and we can make an assignment here. Now, the moment we make an assignment, which means the 1st row, we have made an assignment, the 3rd column also has an assignment. Therefore, the 3rd column, we cannot have an assignment here. Now, that is shown through this. So, this one we cannot have an assignment.

Now, I see, whether I can make an assignment in the second row. Now, the 2nd row has two 0 positions, we made the assignment in the 1st row, because there was only one 0 position, so we made the assignment. Now, there are two 0 positions, so we can either make the assignment here in the first position or we can make the assignment here. But, at the moment, if there are two 0's, we temporarily not make an assignment and proceed.

And the 3rd row also has two 0 positions, and therefore we do not make an assignment to begin with. Then, we go to the 4th row and the 4th row has only one 0, and therefore we can make an assignment. This assignment is shown in the blue color and I am also marking it inside a box. So, in the 4th row, we have made an assignment. So, this is already assigned, so the blue color shows it is already assigned, so I am just making another box over that, it is already assigned.

Now, since the 4th has an assignment in the 2nd column, the 2nd column cannot have one more assignment, and therefore this will also go and that is shown by another X, which is there. So, right now we have made 2 assignments and there are still some 0's that are left. So, we now start looking at it column wise and see, what we do; now we realize that, we had already made assignments here and here already. So, if I look at the 1st column, the 1st column has only one 0, and therefore I can make an assignment here.

Now, when the 1st column has only one 0; the assignment has been made, row 3 has now has an assignment, and therefore we cannot make an assignment here. Now, that is shown by another X; that is coming here. So, now, go back and repeat the 2nd column is already assigned; the 3rd column is already assigned. The 4th column has only one assignable 0, so we make an assignment in this one.

We can also see this and once we put this, now we will have 4 assignments that are shown in blue color. So, we first started with this assignment, then we did this assignment, then we did this assignment and then we did this assignment. So, the assignments are shown in the blue color as well as in the red box. So, now, we have made 4 assignments, we have made assignments only in the 0 positions and every row and every column has 1 assignment, and therefore this is a feasible solution.

Now, this 4 assignments correspond to X 1 3 equal to 1, this position 1st row 3rd column X 1 3 equal to 1, X 2 4 equal to 1, X 3 1 equal to 1 and X 4 2 equal to 1, these are the 4 assignments that we have made. Now, we have got a feasible solution with 0 cost in the reduced matrix, therefore it is optimum to the reduced matrix, and therefore it is optimum to the original matrix. Because, we got the reduced matrix under the assumption that the solution to the reduced matrix is the same as the solution to the original matrix.

So, again to repeat these are the 4 assignments. So, to repeat again just to show the assignments in within the box, this was our first assignment, this went off, this was our second assignment, this was our 3rd assignment and this is our 4th assignment in this order. So, when we made the first assignment, this was eliminated, when we made this second assignment, this one was eliminated. When we made the first assignment, this 0 was eliminated, when we made the second assignment, this was eliminated. When, we made the 3rd assignment here, this was eliminated and then we made the 4th assignment.

Now, this is optimum to the reduced matrix, this will also be optimum to the original matrix. So, the positions, where the assignments have been made, the costs are shown here. So, the cost of the assignment is 4 plus 8 12, 12 plus 9 21, 21 plus 6 27 is the total cost, which is the cost corresponding to the optimum solution. And the optimum solution is given by X 1 3 equal to 1, X 2 4 is equal to 1, X 3 1 is equal to 1 and X 4 2 equal to 1 with objective function value equal to 27.

So, from the first example, what we understood is a very simple way to solve the assignment problem, where we start with a given matrix. We reduce the matrix by subtracting the row minimum from every row, column minimum from every column, such that, we get one 0 at least one 0 in every row and every column. Start making assignments, first start by looking at the rows and then look at the columns and so on.

If there is only one 0, if there is only one assignable 0, then make an assignment which is what we did. We started again to repeat the process, we started with this assignment, because there is only one 0 with, therefore this got automatically cancelled. Now, the 2nd row had at that point two 0’s, so we did not make an assignment. The 3rd one would also have two 0’s, we did not make an assignment, this one has only one 0, we made an assignment and cancel this.

Then, we went back to the column now the column has only one 0, we made this assignment and cancelled this. These 2 columns were already assigned. Now, this had only one assignable 0 which is here and we made the assignment and realize that, we have got 4 assignments, which means we have got a feasible solution and then we gave the answer.

Now, the next question is will this happen every time, we solve an assignment problem or can something else happen? Obviously, something else can happen and we see what can happen, which is very different from what we have seen. We will look at it through another example.

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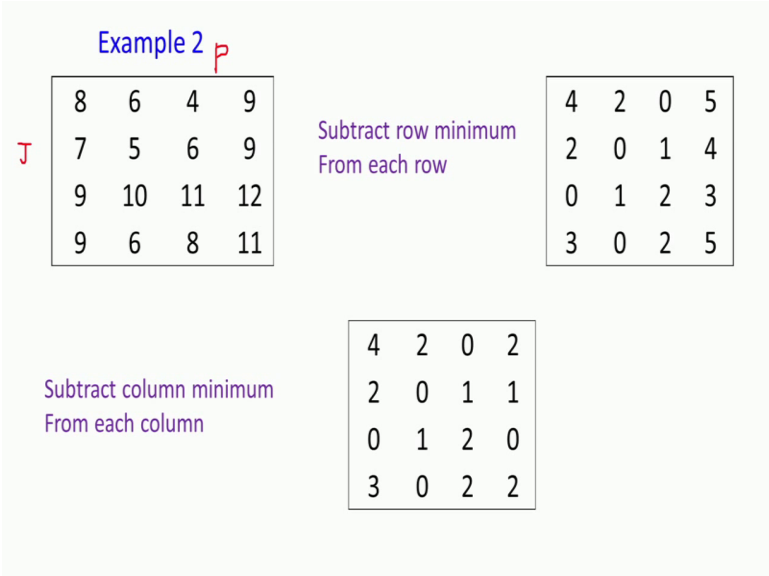

So, we will now go to the second example and try to solve a similar, but different looking assignment problem. So, this is our second example, which is again a 4 by 4 assignment problem with slightly different cost coefficients. So, again we can say that, this represents the four jobs this represents the four people. So, again based on the same two principles, we will first try to subtract the row minimum from every row and column minimum from every column and try to get a matrix that has at least one 0 in every row and every column.

And we will make assignments with such a matrix and see, whether we can get 4 assignments or see, whether we can get a feasible solution that would have 4 assignments. So, that is the method. So, first thing is, we subtract the row minimum from every row, the row minimum in the 1st row is 4. So, we subtract 4 to get 4, 2, 0, 5 and so on. The 2nd row, the row minimum is 5, so it will become 2, 0, 1, 4 and so on, 3rd row the row minimum is 9, the 4th row, the row minimum is 6.

So, we get from the 1st row, we would get subtracting 4, we would get 4, 2, 0, 5; which you can see there. Subtracting 5, you will get 2, 0, 1, 4; subtracting 9 from the every element of the 3rd row, you will get 0, 1, 2, 3 and subtracting 6, you will get 3, 0, 2, 5. Now, we start looking at the columns and subtract the column minimum. The 1st column the minimum is 0, so we retain it, 2nd column the minimum is 0, we retain it, 3rd column the minimum is 0, we retain the column, 4th column the minimum is 3, so we subtract 3 to get 2, 1, 0 and 2.

So, this is the reduced matrix, after we subtract the column minimum from the 4th column. Now, we can start making assignments in this matrix and we will start doing that.

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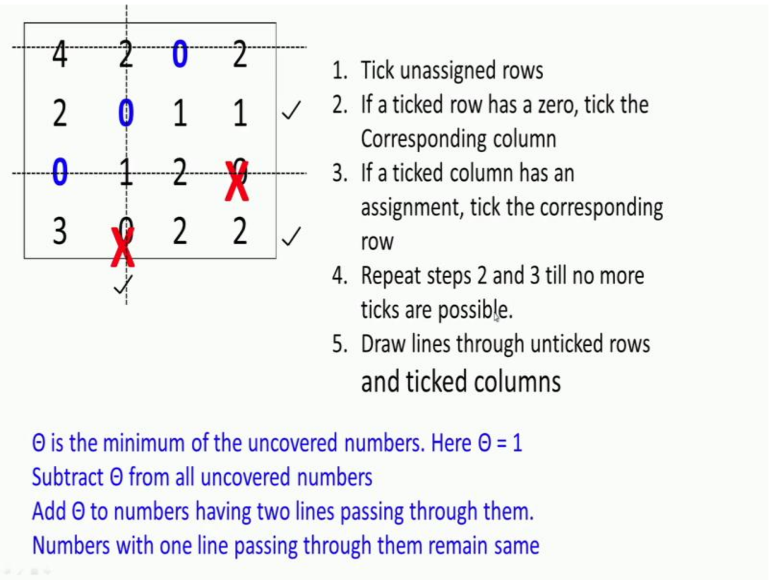

So, I have given the matrix as it is, now we look at the 1st row, the 1st row has only one assignable 0. We make an assignment, which is shown in the blue color and again, let me add the box just to show that, it is an assignment. So, we have made our first assignment and because of this first assignment, there is no other, the 1st row is assigned to the 3rd column, but there are no other 0's here. So, we keep it as it is.

Now, we look at the 2nd row and then we realize we already made an assignment here, first assignment was here, just showing it by a box and now the 2nd row has only one 0, therefore we make another assignment here. So, we have made an assignment here, 2nd row goes to the 2nd column and because, we have made this assignment here. We have to cancel this 0, because the 2nd column can have only 1 assignment and that is given by this red color X, which cancels out this 0.

Now, I go to the 3rd row and I observe that, there are two 0's. So, I would not make an assignment. Now, I and I go to the 4th row and I realize that there is no assignable 0 for the 4th row. Now, I come to the 1st column, there is one 0 here, so I will make an assignment. So, that is going to be shown in the same table here. So, again I will look at mark all the assignments, this was our first assignment, this was our second assignment, which cancelled out this X, and then we look at the 1st column and we make an assignment here.

So, when we make this assignment here, automatically this will go, because by making an assignment in the 3rd row 1st column, the 1st column has an assignment, the 3rd row also has an assignment. So, this 0 is not going to add value and that is again shown with the red one coming. So, now, we go to the 2nd column, it I already assigned, 3rd column is already assigned, the 4th column does not have an assignable 0.

So, if we keep repeating looking at the rows and columns, we realize that we cannot make anymore assignment. We cannot make any more assignment, because all the 0's in this matrix have either been box, which means, have they have been assigned or they are they are crossed, which means they are not fit for assignment. So, we need to stop the assigning procedure at this point and we observe that, we have made only 3 assignments as against 4 that we wish actually have. So, we have not yet got that feasible solution.

Remember that we will make assignments only in 0 positions; we will not make assignments in the non-zero positions. For example, the 4th possibility is this 2. So, we will not make an assignment in this 2 and say that, we have got a solution, no. We will make assignments only in 0 positions, because we are working on a principle that if I make assignments in 0 positions and I get a feasible solution. Then, such a solution is optimum, because the cost associated with that is 0.

So, we need to do something to this matrix to get another matrix and then we need to make assignments in the other matrix. So, what do we do, now we make some changes to this matrix by using a procedure, so let me start that procedure now? So, we would now say that we draw some, we make some ticks and we are going to draw some lines based on an algorithm, so tick an unassigned row.

So, this 4th row does not have an assignment tick an unassigned row now if there is a 0 in that row tick the corresponding column there is a 0 here tick the corresponding column. So, tick the corresponding column. Now, this column has been ticked, if there is an assignment in that column, then tick the corresponding row. So, this row will now have to be ticked. Again, if there is a 0 tick that column, so there is a 0, but the column is already ticked, therefore no more ticking is possible.

So, again let me repeat the procedure, tick an unassigned row, if there is a 0 in a ticked row, then tick the corresponding column, if it is not already ticked. If a column is ticked and if there is an assignment, tick the corresponding row. If it is not already ticked, repeat this procedure, that is the new thing and we will show this here. Tick an unassigned row, if a ticked row has a 0, then tick the corresponding column. If a ticked column has an assignment, then tick the corresponding row. Repeat these two steps, such that no more ticking is possible and draw a lines through unticked rows and ticked columns.

Now, we have drawn these lines, these lines have being drawn through unticked rows, this is an unticked row, this is an unticked row and unticked column, this is a ticked column, this is a ticked column. So, we have drawn a lines through unticked rows and ticked columns. Again, to explain it, tick an un assigned row, this was a our first tick, if there is a 0, tick that column, this was the second tick. If there is an assignment, tick that row, this was a 3rd tick.

Now, repeat this step 2 and 3, no more ticking is possible draw lines through unticked rows and ticked columns. If we do that, we observe something interesting. Now, in this procedure of drawing the lines, we have now drawn three lines and you will realize that there are 3 assignments shown in blue color, there are 3 assignments and we have drawn three lines.

Not only that, the three line will cover all the 0's, all the 0's in the matrix will have at least one line passing through them. They may have non-zero also covered, but the important thing is every 0 is now covered by one line. So, we have now made that ticks and we have drawn the lines. Now, we have to do something with this to generate a new matrix with which we will actually be solving this problem. Now, what do we do after this ticking and how do we modify this to get into a new matrix, we will see in the next class.