Introduction to Operations Research
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Module - 07
Assignment Problem
Lecture - 01
Introducing the Assignment Problem

In this module, we discuss the Assignment Problem. The assignment problem is another important problem in operations research, like the transportation problem. Let me define the assignment problem.

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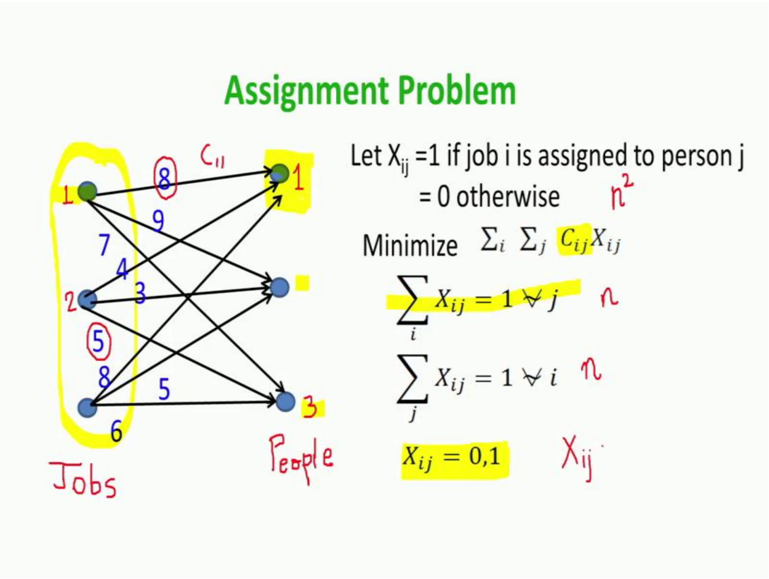

Now, in the assignment problem, let us say there are three jobs, which can be done by three people. The graph or the network that is shown here represents the problem. So, we will assume that, there are three jobs that can be done by three people. So, we say these three represent the jobs and these represent people. So, these represent people, who can do these jobs. Now, this to carry out these jobs, it is going to cost something to allocate these jobs to these people. For example, this figure 8 would be the cost of giving job 1 to person number 1.

So, job 1 to person number 1, it costs say 8 rupees or 8 units of money. For example, it costs 5 rupees or 5 units of money to allocate job number 2 to person number 3. So, there are 9 such costs, there is a cost that each person is going to charge or it is going to cost the organization to allocate these jobs to people. There are an equal number of jobs and people.

So, at the end these jobs have to be allocated to people, such that, each person gets only one job and each job goes to only one person, each person gets only one job. There is a one on one assignment. How to do that at minimum cost is called the assignment problem. Looking at the problem from a slightly different perspective, since we have three jobs that can be assigned to three people and in general, if there are n jobs that have to be assigned to n people.

This can be done in n factorial ways, because the first job can go to any one of the three persons, if it is a 3 by 3 problem. The second job will go to one, can go to any one of the remaining 2 because one of them would get the first job and the third job will go to the third person. So, if there are three jobs, it can be done in three factorial ways and if there are n jobs in general, it can be done in n factorial ways.

Now, among these three factorial possibilities in this problem or n factorial in general, which one has the least cost? How do we find that one out of the n factorial that has the least cost is the assignment problem? It is very important to note that in the assignment problem, we have an equal number of jobs and people. It need not always be jobs and people; it can be jobs and machines, machines and people, projects and students, many ways.

But, an equal number of two different things, and then we have to assign one to the other. At the end of it, every job goes to only one person and every person gets only one job. Now, let us formulate this problem, let X i j be equal to 1, if job i is assigned to person j, it is equal to 0 otherwise. Now, here we observe that the definition or the way the decision variable is defined is a little different.

Earlier, we had defined X i j to be a continuous variable, it can take any non negative value. Now, this X i j can take only two values 1 or 0, so it is a binary variable. In fact, if you see carefully, we have seen a formulation very early in this course, where the decision variable took the binary value. So, assignment problem, we begin by saying that X i j is a binary variable that takes either 1 or it takes 0.

Now, what is the objective? The objective is to minimize the total cost of assignment or total cost of allocation. So, that is given by summed over i summed over j, C i j, X i j, where C i j is the cost of allocating i to j. So, in general a cost is here it is C 1 1 or in general, it is C i j. The cost of allocating is C i j. So, we want to make these allocations in such a way that the total cost of allocation is minimized.

Now, what are the constraints? There are two sets of constraints. So, one set of constraint is sigma X i j equal to 1 summed over j for every i. Now, here if we look at this network, this part is the i, the jobs. So, this part is the i which are the jobs. So, if I take i, for example, if I take the first job, which is here, then this first job can be assigned to either the first person or the second person or the third person. So, it can go to only one person, one job will go to only one person. So, summed over i X i j equal to 1.

So, this person, if I take the first person j equal to 1, this person gets only one job, this person will get either this job or this job or this job, summed over i X i j is equal to 1 for every j. Second set of constraint is summed over j X i j equal to 1 for every i, which means, if I take i equal to 1, if I take this job and this job will go to either this person or this person or this person.

The first constraint will say that this person will get either this job or this job or this job. The second constraint will say that this job will go to either this person or this person or this person. So, first set is about a person getting only one job and the second set is about a job going to only one person. So, there are three constraints, if you look at a 3 by 3 assignment problem. In general there are n constraints here and there are another n constraints here.

So, we will see that there are n constraints here, there are another n constraints here and there are n square variables, decision variables, because i equal to 1 to n j equal to 1 to n gives us n square decision variables and 2 n constraints. The n squared decision variables are binary variables, so we define X i j as 0 or 1, so they are defined as binary variables with 0 or 1.

Now, this is the formulation of the assignment problem, if you elaborate and go to the first constraint, X i j equal to 1 summed over i for every j. Suppose I take this j, j equal to 1, the constraint will be X 1 1 plus X 2 1 plus X 3 1 is equal to 1. For this, X 1 2 plus X 2 2 plus X 3 2 equal to 1 and for this X 1 3 plus X 2 3 plus X 3 3 is equal to 1. If I take this set of constraint for i equal to 1 this will be X 1 1 plus X 1 2 plus X 1 3 is equal to 1 and so on. So, there will be six constraints and nine variables in a 3 by 3 assignment problem.

And in general in a n by n assignment problem, there are n square decision variables and 2 n constraints in a n by n assignment problem. All the decision variables are binary variables. Now, if these X i j's are not defined as binary variables and if they are defined as continuous variables, then the assignment problem is a linear programming problem. For example, if I replace this X i j binary as X i j greater than or equal to 0.

If I replace these binary variables by continuous variables, the assignment problem is a linear programming problem, because the objective function is a linear function, the constraints are linear and there is a non negativity restriction. So, if we replace this and make it a linear problem, then we can use either the simplex algorithm in it is tabular form or any other linear programming way to solve the assignment problem.

It is also important to note the assignment problem, if we define this X i j to be a continuous variable is also like a transportation problem, where there are an equal number of supply points and demand points. And each supply points supplies 1 unit and each demand point requires 1 unit. So, the assignment problem can also be seen as a special case of a transportation problem, where there are equal number of supply points and demand points. Each supply points supplies only 1 unit and each demand point requires only 1 unit.

So, the assignment problem is not only a linear programming problem, it is also a transportation problem. We have already seen methods to solve linear programming; we have already seen methods to solve transportation and transportation solution methods are slightly different from the linear programming solution method. It is also interesting to note that, we solve the assignment problem using a different algorithm, which is very different from solving a transportation problem or solving a linear programming problem.

Even though, the assignments problem is in principle a linear programming problem, it is also in principle a transportation problem. We do not use those methods to solve it, we look at solving the assignment problem by a special method, which is faster and which is tailor made to solve the assignment problem. Now, we will look at ways of solving the assignment problem now.

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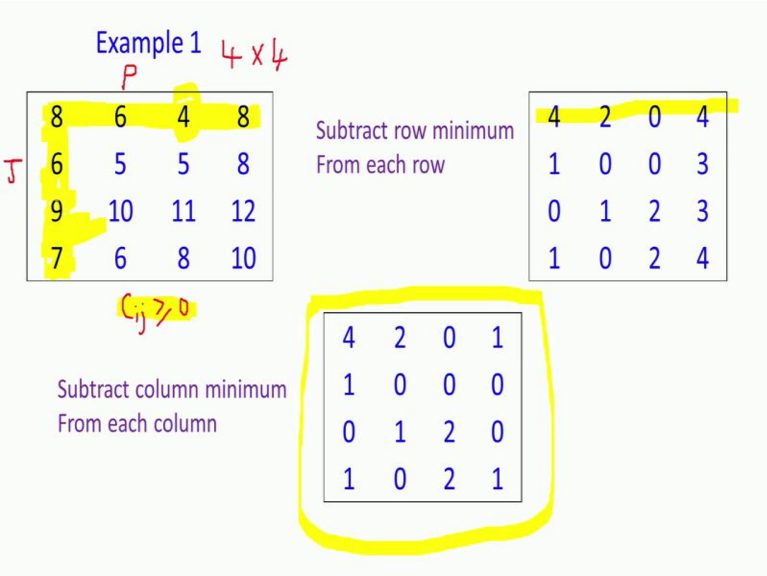

Now, we look at the first example, it is customary to write the assignment problem in the form of a table as shown here. So, this 4 by 4 table represents a 4 by 4 assignment problem. So, it is a 4 into 4 assignment problem, we can assume that, these 4 are jobs and these four are people. So, we would say that, it costs 8 to allocate the first job to the first person; it costs 6 to allocate the second job to the first person and so on. So, it is a 4 by 4 assignment problem.

It is also possible to understand that, because this assignment problem has an equal number of jobs and people, the matrix is always a square matrix. If it is a 4 by 4 problem, it is a 4 by 4 matrix and so on. So, whether we solve this matrix or whether we solve it is transpose, it means, if we assume that instead of 8, 6, 4, 8 as rows 8, 6, 4, 8 becomes the columns and so on, it would still give the same solution. But, right now let us look at defining clearly that the 4 rows represent the four jobs and the 4 columns, the four people and then let us solve the solve the assignment problem.

Now, the assignment problem is the solution to the assignment problem is based on two simple principles. The first principle is, if the cost coefficient matrix like the matrix that we have seen here or we have shown here. If this cost coefficient matrix also called the C i j matrix, it is C i j matrix, which is shown here is greater than or equal to 0, which means, it is going to cost us either 0 or some other cost to allocate a job to a person. If we are able to get a feasible solution with 0 costs, then such a solution has to be optimum.

The reason is all the cost coefficients are non negative and if for some reason, we are able to get a feasible solution with 0 costs, then the objective function value is 0. Therefore, it is optimal, but at the moment, we do not have any cost any of these 16 cost elements as 0 and all these 16 costs are positive. Therefore, what we do is, we try to reduce this matrix to another matrix, which has 0 costs and in the process; we try to make assignments only in the 0 positions and try to get a feasible solution.

Now, what is a feasible solution? A feasible solution is one that satisfies all the constraints. In this case, it is a 4 by 4 problem, there will be eight constraints, each job will go to only one person, each person will get only one job, which means, every row should have only one assignment and every column should have only one assignment. If we are able to get that, then it is a feasible solution.

So, again let me go back, we start with the principle that, if we are able to get a feasible solution with 0 costs, then it is optimum, but then we observe that, there are no costs that are 0. So, we try to reduce the 4 by 4 matrix into another matrix, which would have 0's and then make the assignments only in the 0 positions and try to get a feasible solution. Then, such a feasible solution will be optimum, because it will have 0 costs. Now, how do we reduce this matrix to another matrix which has 0's, now that is based on the second principle, the second is this.

Suppose, I take the first job, the first job can go to any one of these four people and the first job can go to either the person 1 or person 2 or person 3 or person 4, because it has to go to one out of these four people. If all of them reduce the corresponding cost by 1 unit, it will still go to the same person, because all of them had reduced it by 1 unit. Therefore, subtracting a constant from every element of the row will not change the solution.

In a similar manner, if I take these four jobs the cost of allocating these four jobs to the first person. If we reduce the cost all these 4 costs by 1 unit each, this person would still get the same job allotted to him or her, which also means that, subtracting the same constant from every element in a column will not affect the solution. So, the second principle is subtracting a constant from every element in a row will not affect the solution, subtracting the same constant from every element in a column will not affect the solution.

Again, solution means values that the decision variables takes and in this case, the actual allocation of jobs to persons. So, if we take the first row and we realize that, if we subtract the same constant from every element of the first row, the solution is not getting affected. So, now, let us try and subtract some constant and try to subtract as much as we can. So, the maximum that we can subtract from every element of the row is the minimum of the 4 costs, which is 4, because up to subtracting 4 the C i j will remain non negative.

If I subtract 5 from every element of the first row, then this thing will become minus 1 and I do not want to work with the matrix, where a C i j is negative. So, we have to make sure that C i j's are greater than or equal to 0. Therefore, if we subtract, we subtract as much as we can and in the process, we want 0's. Therefore, we subtract the row minimum from element of the row, so that after subtraction this row will have at least 1 0.

So, we subtract 4 from every element of the 1st row, we subtract 5 from every element of the 2nd row, we subtract 9 from every element of the 3rd row and we subtract 6 from every element of the 4th row. So, in principle, we subtract the row minimum from every element of each row and if we do that, we get a new matrix, which is like this. So, subtracting 4 from every element of the first row would give us 8 minus 4, 4, 6 minus 4 2, 4 minus 4, 0 and 8 minus 4, 4, which are the four values that we see from here.

Similarly, we subtract the row minimum from the second row to get 1, 0, 0, 3, we subtract the row minimum from the 3rd row to get 0, 1, 2, 3 and we subtract the row minimum of 6 from the 4th row to get 1, 0, 2 and 4. So, now, we have reduced this matrix to the one on the right hand side with a very interesting addition that every row now has at least 1 0. You will realize that the second row has 2 0's, because 5 was the minimum and it occurred in two different places. Now, every row has 1 0.

Now, what is applicable to the row is also applicable to the column, because subtracting the same constant from every element of the column will not change the solution. So, we now look at the first column, the column minimum is 0, so when we subtract, we will get the same thing. We look at the second column the column minimum is 0 and we will get the same numbers.

The 3rd column also has column minimum is equal to 0, so we will get the same numbers. But, the 4th column has column minimum equal to 3. So, we subtract the column minimum from every element of that subtract the column minimum to get a new matrix, where the first 3 columns are the same. The last column has changed to 1, 0, 0 and 1.

So, we have now reduced the first matrix to this new matrix, which is here, we have reduced it to this matrix, which is here row. And then if we solve this matrix the solution will be the same as that of the first one. How we solve this matrix, we will see this in the next last class.