Introduction to Operations Research
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Module - 06
Transportation Problem
Lecture - 05
MODI Method; Dual of the Transportation Problem and the Optimality of the MODI Method

In this class, we continue the discussion on the MOdified DIstribution Method.

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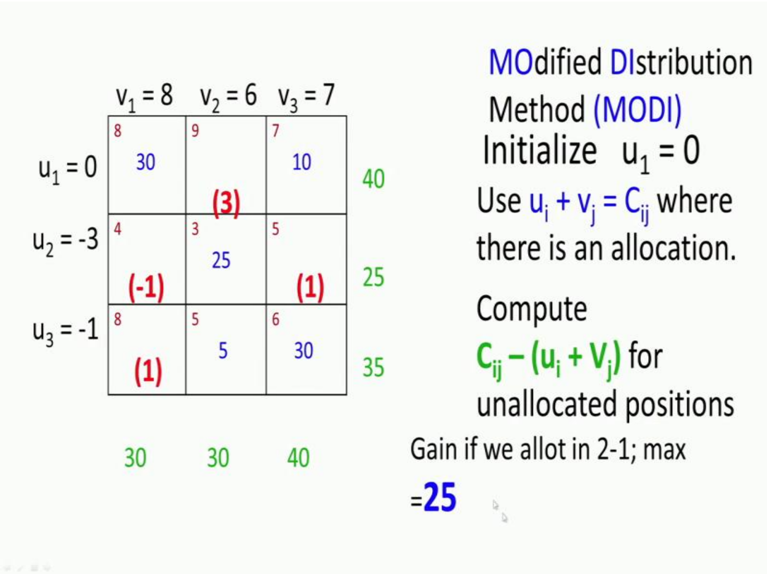

We started with a solution which had 30 allocated to this position 10, 25, 5 and 30. Now, this solution is feasible, and using this solution we also defined some u's and v's. So, we started with a solution, and then we defined some u's and v's, we started with u 1 equal to 0, and then we used u i plus v j is equal to C i j, where there is an allocation. So, using this u 1 equal to 0 we got v 1 equal to 8, because there is an allocation.

So, using this allocation we got v 3 and once v 3 is known, we use this to get u 3 and once u 3 was calculated we used this allocation and this cost to get v 2 and once v 2 was calculated we used this cost to get u 2, so all the u's and v's are computed. Once, the u's and v's are computed we also said that we will compute C i j minus u i plus v j for the unallocated positions and when we did that for this position the value is 3, 1 minus 1 and 1.

Now, this minus 1 tells us that it is possible to put something in that position and gain. So, when C i j minus u i plus v j becomes negative for the unallocated position, we understand that we can put something there and we can reduce the cost further. You will also observe that these four values that are shown in the red color bold are the values that we obtained when we actually applied this stepping stone. When we had put a plus 1 here and completed the loop and found out the additional cost we said the cost would be 3 for every additional item that we put here.

And if you put one item here and complete the loop, you will realize that there is a gain which is given by this minus 1. So, we know that if you put one unit there is a gain of 1 rupee or 1 unit gain there, what is the maximum I can put there is the next question.

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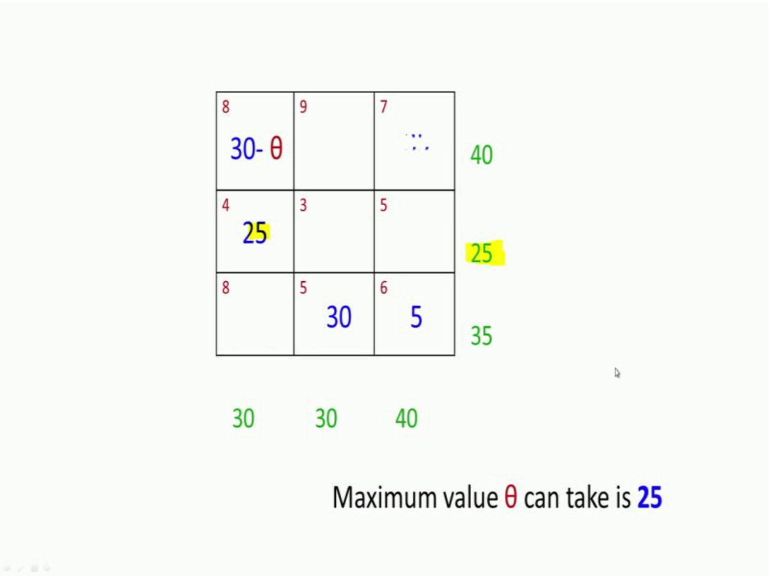

So, to do that we go back and look at this, the same solution is shown and then I say that I am going to put a theta in this position. Now, when I complete the loop, now this becomes 25 minus theta, so that this 25 is satisfied, so that this 25 is balanced and taken care of. Now, that I have put a minus theta here I should put a 5 plus theta, so that that is balanced, so the 5 goes and becomes 5 plus theta. And once again, since this is balanced now, but we have increased a theta here, so 30 has to become 30 minus theta.

And since we have put a minus theta here, once again to balance this we will get 10 becomes 10 plus theta and this 30 will finally, become 30 minus theta. Now, when we look at all these as we increase theta in this position we observe that 30 minus theta is reducing, 25 minus theta is reducing and 30 minus theta is reducing and the best value that theta can take is 25 beyond which this will become negative.

Therefore, we put theta equal to 25 and we redo this allocation and if we redo this allocation this becomes 25, 25 minus theta goes 5 plus theta becomes 30, 30 minus theta becomes 5, 10 plus theta becomes 35 and 30 minus theta becomes 5, so this is a new solution that we have obtained. Now, we need to check whether this solution is the best solution or whether some more gain is possible. To do that we once again initialize u 1 equal to 0 and repeat the procedure.

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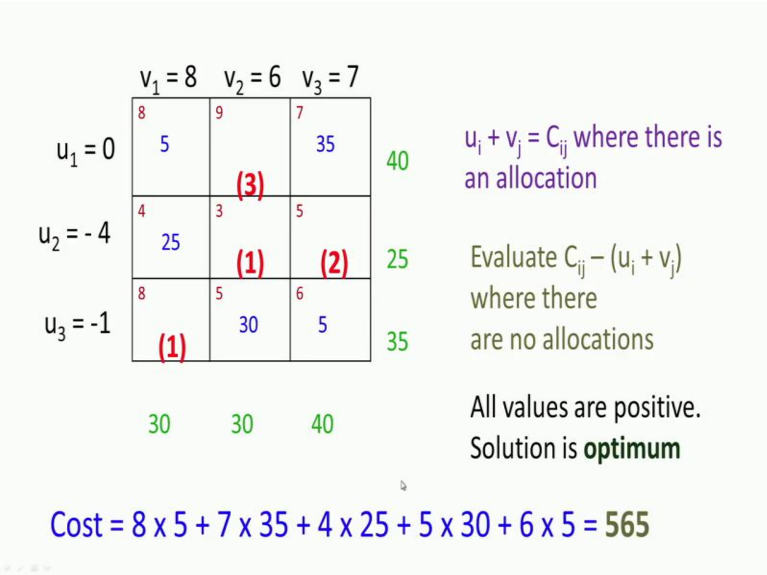

So, we initialize now I have shown the same solution here, now we initialize u 1 equal to 0, now we go back and observe that here there is an allocation. So, u i plus v j is equal to C i j, therefore v 1 is equal to 8. Now, with v 1 equal to 8 there is an allocation here, so v 1 plus u 2 is equal to 4, so u 2 will become minus 4. Now, again with this there is an allocation here, so u 1 plus v 3 is equal to 7, so v 3 becomes 7, now there is an allocation here, so u 3 plus v 3 is equal to 6, so u 3 will become minus 1. Now, u 3 plus v 2 is equal to 5, so v 2 will become 6, so now, the u's and v's are calculated.

Now, we go back and evaluate C i j minus u i plus v j where there are no allocations or in the unallocated position. So, for this position 9 minus 6 plus 0 which will be 3, for this position 3 minus 6 minus 4 2; therefore 1, for this position 5 minus 7 minus 4 3 which is

2 and for this position 8 minus 1 7 which is 1. Now, this value shown in red are all positive which means putting a unit in these positions is not going to reduce the cost.

Therefore, we stop the algorithm and say that we do not have a way by which we can reduce it further, and therefore the algorithm stops. So, this is how the Modified Distribution algorithm or MODI algorithm works giving us the optimum solution and the cost is 565 which is 8 into 5 plus 7 into 35 plus 4 into 25 plus 5 into 30 plus 6 into 5.

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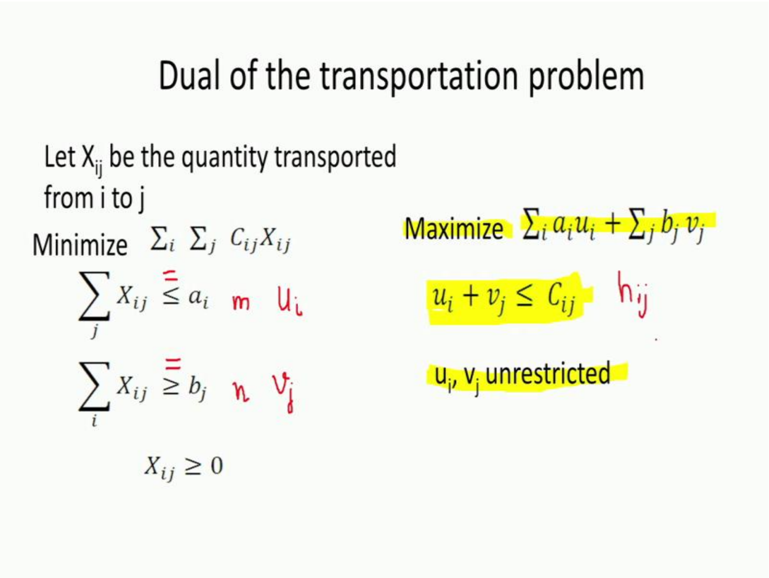

Now, so far we have seen two methods the stepping stone method and the modified distribution method that help us get the optimum solution from a given starting solution. There are still some issues involved, the starting solution has to be basic feasible and so on. But, for the sake of this course we will assume that we would use either the min cost method or the Vogel's approximation method to get a good starting solution and these starting solutions have m plus n minus 1 allocations, if there are m supply points and n demand points.

And these allocations are independent and we also know that the minimum cost method and the Vogel's approximation method give us such solutions. So, now, we have to understand why the MODI method gives the optimum solution. In earlier classes we have spent a lot of time understanding the dual and also in writing the dual. So, let us now try and write the dual of the transportation problem, now what is shown in the left is the primal, where X i j is the quantity transported from i to j and we have shown the unbalanced problem, where what is taken should be less than or equal to the supply and what is sent to the demand points should exceed or meet the demand.

We also studied the balanced version, where we said that these constraints are equations when the total supply is equal to the total demand. So, we look at the balanced problem and then we write the dual of the balanced problem. We also said that if there are m supply points, there are m constraints here and if there are n demand points there are n constraints here. So, we now introduce dual variable, now these dual variables are called u i's which are u 1 to u m, for the m constraints and they are called v j for the j demand points.

So, in our 3 by 3 example we will have 3 u i's and 3 v j's., so if we define these then if we write the dual, then we know that the objective function of the dual is the right hand side coefficient multiplied by the dual variables. So, the dual will be a maximization problem, because the primal is a minimization problem and the a i u i b j v j summed over i and j is the objective function. There will be as many dual constraints as the primal variables, so there are m into n primal variables, there will be m into n dual constraints.

So, if we take a typical X i j that X i j will appear in the i'th constraint here and in the j'th constraint in this. So, the corresponding dual variables will be u i and v j, so when we write the dual that constraint will become u i plus v j is less than or equal to C i j which is the objective function coefficient of the value corresponding to X i j. So, the typical dual constraint is u i plus v j is less than or equal to C i j and there will be as many dual constraints as the number of primal variables. In our example, we solved a 3 by 3 problem, so there will be 9 dual constraints for each i j.

Importantly these u i's and v j's are unrestricted in sign, because we have now assumed a balanced problem, the primal constraints are equations. Therefore, the double variables will be unrestricted in sign, we have seen this before that if the primal constraint is an equation, then the dual variable is unrestricted in sign, so this is the dual of the transportation problem. Now, let us understand what we do under the MODI method by looking at the dual of this problem ((Refer Time: 12:02)).

Now, this is a situation when we had a solution like this 30, 10, 25, 5 and 30 and we initialized u 1 equal to 0 and then we used u i plus v j is equal to C i j wherever there is an allocation and we found out the 5 remaining values, we initialized u 1 we were able to find out v 1, v 2, v 3 and u 2, u 3. Because, there were five allocations m plus n minus 1 3 plus 3 minus 1 five allocations and these five allocations are sufficiently independent for us to give the remaining five values.

Now, with every position for example, now let us take this position where there is an allocation. Now, here is a position where there is an allocation which means in this position X 1 1 is 30 this is a position 1 comma 1, first supply, first destination 1 comma 1, so X 1 1 is 30. Now, we also know that wherever there is an allocation u i plus v j is equal to C i j that is how the u i's and v j's have been computed.

So, now, go back with this understanding let us go back to this. So, what we have done the way we have calculated the u i's and the v j's is when there is an allocation, when X i j is basic or when X i j is in the solution when there is an allocation for that position u i plus v j is equal to C i j. Now, if we take a typical constraint u i plus v j is equal to C i j or if we take the first constraint this would mean u 1 plus v 1 is less than or equal to C 1 1 is the constraint.

Now, X 1 1 is in the solution ((Refer Time: 14:21)) and we now observe that the next 1 1 is in the solution u 1 plus v 1 is equal to C 1 1 which means if u i plus v j less than equal to C i j is converted to an equation we will be adding a slack variable. Now, let that slack variable be called h i j, let some h i j be the slack variable. Now, when u 1 plus v 1 is equal to C 1 1 then the corresponding slack variable is 0. So, whenever u i plus v j is equal to C i j, the corresponding slack variable is 0.

Now, we also know that whenever X i j is in the solution u i plus v j is equal to C i j, which means when X i j is in the solution then the corresponding dual constraint the slack is 0, which means x v is equal to 0 in our notation which also means complimentary slackness is satisfied. So, whenever there is a variable in the solution, we observe that the corresponding slack is 0, the constraint u i plus v j becomes C i j, therefore the inequality becomes an equation which means the corresponding slack is 0 and complimentary slackness is satisfied.

Now, for all the other places we are evaluating C i j minus u i plus v j, which means we are essentially trying to calculate the slack. Because, we also said that if this is a constraint, if this inequality is written as an equation u i plus v j plus h i j is equal to C i j. So, h i j will become C i j minus u i plus v j, ((Refer Time: 16:33)) so what we are doing is in all these four positions we are trying to find out the h i j, we are trying to find out the dual slack.

If the dual slack is positive, it means the dual constraint is satisfied, if the dual slack is negative the dual constraint is not satisfied. So, here the dual constraint is not satisfied the dual is infeasible, we also know that when the dual is infeasible and complimentary slackness is applied the primal is non optimum. So, we try to make that dual feasible by bringing these into the solution and by making an allocation here, which is what we did and then we got a new solution and then we repeat this procedure till there all the h i j's are positive or till all C i j minus u i plus v j are positive.

Now, it means if I have an allocation for which all C i j minus u i plus v j's are positive, it means I have a feasible solution to the primal, complimentary slackness is satisfied and now I have a corresponding feasible solution to the dual, which means the optimum solution has been reached. So, the MODI method even though we do not do this every time, we only do the calculation we define the u i's and the v j's this way and then we calculate C i j minus u i plus v j and then when we stop ((Refer Time: 18:09)) when all this C i j minus u i plus v j's are positive saying that the optimum is reached.

What we are actually doing is, for a primal solution primal we are evaluating, the dual the u's and the v's are the dual variables. We are unduly worried about the fact that this is negative, because the duals can be unrestricted in sign they can take a negative value, so we are not worried about that. So, we have a feasible solution to the primal we maintain complimentary slackness by making sure that wherever there is an allocation u i plus v j is equal to C i j and the slack is 0.

And then we evaluate the slack for places where there are no allocations, if all the slacks are positive then all the dual constraints are satisfied and the dual has a feasible solution which is actually the optimum solution to this problem. When one constraint is not satisfied, which means one of the dual slacks is negative, which means C i j plus minus u i plus v j is negative then we know that the optimum has not been reached we put that is a place where we can that can come into the solution. So, we change the allocations and then repeat this procedure.

So, the MODI method is actually a very good application of the duality principle ((Refer Time: 19:30)). So, with the amount of time we have spent in this course to understand duality, you can immediately see the application in the algorithm that optimizes or finds the best solution to a transportation problem.

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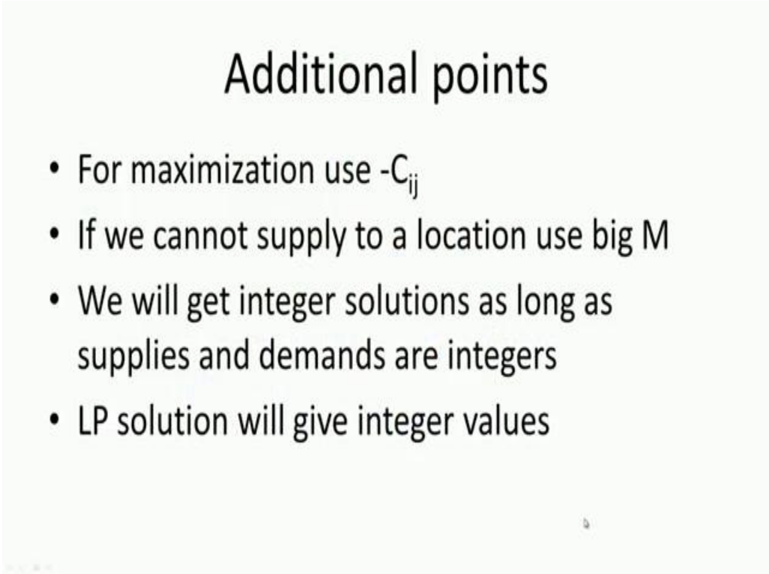

Now, we also have a few additional points when it comes to a transportation problem, usually the transportation problem is a minimization problem, we try to minimize the cost of transportation. But, sometimes we can formulate transportation problems out of some practical situations, where we actually maximize C i j, X i j. So, if we are solving a maximization problem then the best thing to do is to put a negative value to these profits. So, that they notionally becomes cost, so that maximizing profit becomes minimizing cost and work with the negative values that is often suggested if you are solving a maximization problem.

Now, there can be situations where we will say that we cannot transport from a particular supply point to a particular demand point, assuming it is a minimization problem. So, one of the ways to model is to put a big m, a large cost of transporting from that place to the corresponding destination, so that because of the large cost that we have put in, we will not send anything from that supply point to that destination point.

The way we have solved the transportation problem and the way we obtained our starting solution and then from the starting solution to the optimum solution the two stage right through we had integer values for the x i j's or the allocations. So, we will get integer solutions as long as the supplies and demands are integers. In our example, the supply quantities and the demand quantities were integers, and therefore we got integer solutions.

The last point is, if we did not use this combination of say a penalty cost method followed by MODI or a min cost method followed by an MODI or stepping stone and we had simply solved the LP by putting it into a solver or through simplex, then we would still get integer values that is because of what is called the unimodularity property of the matrix, we are not going to go deeper into that idea. But, we will also understand that there are a class of linear programming problems, where even if we define these variables as continuous variables we will end up getting integer solutions.

So, this brings us to the end of this module which is a discussion on the transportation problem, we will now look at the assignment problem in the next module which we will start from the next class.