Introduction to Operations Research
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Module - 05
Lecture - 25
Matrix method for LP problems

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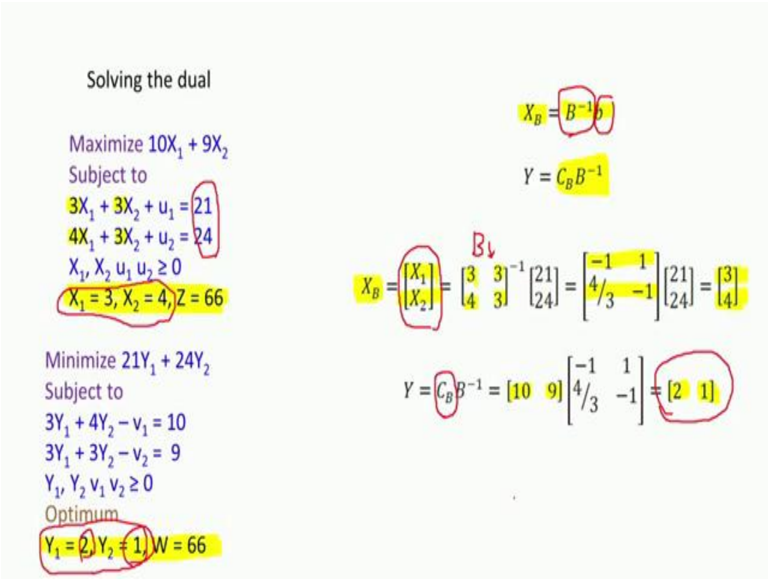

In this class, will see some aspects of solving the dual I will also introduce some matrix notation more in tune with solving this problem and also to introduce the matrix way of solving linear programming problems. Now again we go back to the familiar problem we have maximize 10 x 1 plus 9 x 2 subject 3 x 1 plus 3 x 2 is less than or equal to 21 and 4r x 1 plus 3 x 2 less than or equal to 24. We already know that the optimum solutions is x 1 equal to 3 x 2equal to 4 with z equal to 66.

Now, here is the dual which we have seen and we know that the optimum solution to the dual is y 1 equal to 2 y to equal to 1 with w equal to 66. Now, let us introduce some matrix notation just to show this. There one can spend a lot more time starting from the beginning but, let us just see a little bit of the matrix method. Now, let us assume that I am solving for x 1 and x 2 x 1 and x 2 forms what is called basis forms x b which is a set of basic variables the set of basic variables is called basis let us say x 1 x 2 is now the set of basic variables. So, if you are solving for any x b than if you remember doing some matrix methods of solving equations then the solution to equations is given as x b is a inverse b where a is the coefficient matrix in our case we use notation capital B so, x b is equal to b inverse b.

Now, the solution if I know the basic variables in the primal. If I know the basic variables that I am solving for than x b is equal to b inverse b I will tell you what capital b matrixes is and its inverse the corresponding solution to the dual for the given basis is given by c b b inverse. Let me show all this computation by treating x 1 and x 2 as our basic variables. So, x b is x 1 x 2 now this capital b is called the basis matrix whose inverse I am marking the capital b is the basis. Now, what is the basis? The basis is if x 1 and x 2 are the basic variables the coefficients of x 1, 3 and 4. Now, 3 and 4 will come as the first column of b x 2 is the second basic variable and remember there are two equations therefore, we can solve only for two basic variables. So, x 2 is the second basic variable 3 and three are here. So, 3 and 3 is also here that is your matrix b. So, this is your matrix b. So, b is given by this so, this inverse b inverse. So, b inverse b this small b which is shown here small b is your right hand side So, now, what you can do is you can take this, invert this, multiply with the right hand side. Now, you could have you could have done some methods for matrix inversion the first method that you would have learnt to invert a matrix is to find the determinant and the Co factor and the ad joint and the ad joint b by determinant b is b inverse. Now, b inverse is inverse matrix for a given square matrix with a non 0 determinant.

So, if b is the given square matrix b inverse is another square matrix of the same size such that b into b inverse are b cross b inverse is I b inverse b cross b is also equal to I. So, that is the definition of b inverse. So, you could find the determinant, co factor and the ad joint and if you did that you are your b inverse matrix will look like. This is your b inverse matrix now, b inverse into b right hand side 21, 24 will give you through matrix multiplication minus 1 into 21 plus 1 into 24 which is 3 and 4 by 3 into 21 which is 28 minus 1 into 24 is 4. So, the solution is x 1 equal to 3 x 2 equal to 4 which is your optimum solution for the basis x 1 x 2. Now, what is the value of the dual here by solving the dual we got this solution by solving the dual optimally we got y 1 equal to 2 y 2 equal to 1. Now, the dual is given by c b d inverse what is c b; c b is the objective function coefficient for x b. So, the objective function coefficient for x 1 x 2 is 10 and 9e. So, c b is 10 and 9 b inverse is the same b inverse you could get b inverted to get this and now when we do the matrix multiplication 10 into minus 1 plus 9 into 4 by 39 into 4 by

336 by 3 which is 12 minus 10 plus 12 is 2 and 10 into 1 plus 9 into minus 1 is 1. So, the optimum solution to the dual is 2 and 1 with y 1 equal to 2 y 2 equal to 1 which is y 1 equal to 2 and y 2 equal 1. So, we could do this for us it is also possible to show that simplex does exactly that and we will see how simplex does exactly that.

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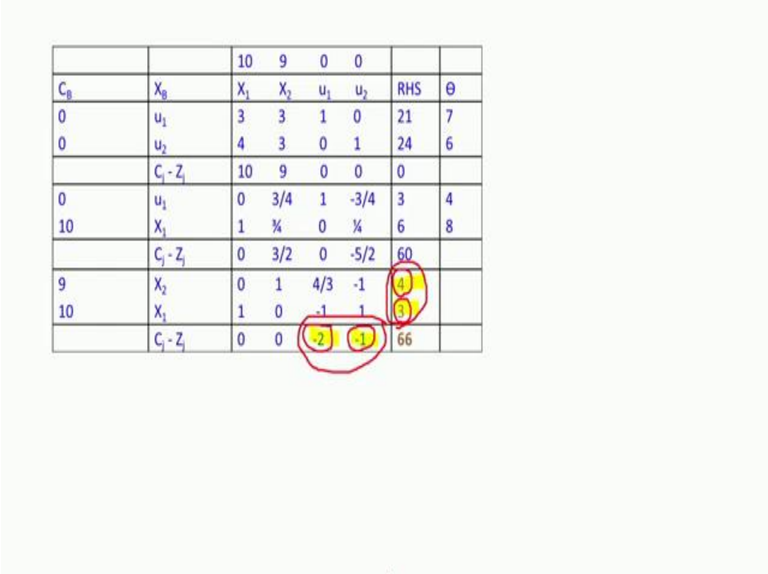

		10	9	0	0		
C_B	X_B	X_1	X_2	u_1	u_2	RHS	θ
0	u_1	3	3	1	0	21	7
0	u_2	4	3	0	1	24	6
	$C_j - Z_j$	10	9	0	0	0	
0	u_1	0	3/4	1	-3/4	3	4
10	X_1	1	¾	0	¼	6	8
	$C_j - Z_j$	0	3/2	0	-5/2	60	
9	X_2	0	1	4/3	-1	4	
10	X_1	1	0	-1	1	3	
	$C_j - Z_j$	0	0	-2	-1	66	

So, we now go back to show the simplex table and then we realize that x 1 x 2 are the shown the other way and now we realize x 1 equals to 3 x 2 equal to 4 y 1 equal to 2 y 2 equal to 1 and then we realize that since we did x 1 x 2. Now, the b inverse that we calculated from here is minus 1 4 by 3 1 minus 1 and now you realize it is minus 1 4 by 3 1 minus one it shown the other way the reason being x 2 is coming first. If I had taken x 2 x 1 as basis which means if I started with 3, 3, 3, 4 then the b inverse would be here since I started with 3, 4, 3, 3 the b inverse first column of b inverse is here and the second column of b inverse is here.

Now, as we already know that the primal solution is shown here and the dual solution is shown here. Now, we also know that this primal solution is b inverse b and we can also show that this dual solution is c b b inverse. Now, in the previous slid we multiplied ten and nine because I had taken x 1 first and x 2. If I take a x 2 first and x 1 second I will multiply with 9 and 10 which is what we actually did. C b b inverse if you see this value the negative will be 0 minus 9 into 4 by 310 into minus 1. So, this is one column of the b inverse multiplied by c b. We got here this is other column of the b inverse multiplied by c b we got here. So, what is simplex does is b inverse b and c b b inverse through our row column operations therefore, we are able to see the solution to the primal as well as solution to the dual in the simplex table we are able to see as the negative of the c j minus z j.

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Minimize $7X_1 + 5X_2$

Subject to

$X_1 + X_2 \geq 4$

$5X_1 + 2X_2 \geq 10$

$X_1, X_2 \geq 0$

$$X_B = B^{-1}b$$

$$Y = C_B B^{-1}$$

$$X_B = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{bmatrix} \begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 10/3 \end{bmatrix}$$

$$Y = C_B B^{-1} = [7 \quad 5] \begin{bmatrix} -2/3 & 1/3 \\ 5/3 & -1/3 \end{bmatrix} = [11/3 \quad 2/3]$$

Now, let us take another example and see how we can show this. Now, let us take the minimization problem that we saw very early here when we did the simplex algorithm. The problem is to minimize 7 x 1 plus 5 x 2 subject to x 1 plus x 2 greater than or equal to 4 subject to 5 x 1 plus 2 x 2 greater than r equal to 10 x 1 x 2 greater than or equal to 0. Once again if x 1 x 2 happened to be the basis than the b matrix x b is b inverse b then the b matrix are the columns corresponding to the basis. So, x 1 x 2 in that order so one and 5 1 and 2. So, I have written 1 and 5 and I have written 1 and 2.

Now, we have to invert this once again we have to find the determinant co factor ad joint or any other way to invert this matrix determinant is 2 minus 5 is minus 3 and so, on. So, we calculate the determinant we will get b inverse we calculate the inverse we will get b inverse this way. So, x is equal to b inverse b so, this multiplied by the right hand side 4 10 would give us would give us minus 2 by 3 into 4 plus 10 by 3. So, minus 8 by 3 plus 10 by 3 is 2 by 2 by 3 and 5 by 3 into 420 by 3 minus 1 by 3 into 10 10 by 3 is10 by 3. So, this is the solution to the primal solution to the dual will be c b b inverse c b is the objective function coefficient of x 1 x 2 which is 7 and 5. So, 7 and 5 multiplied by b inverse minus 14 by 3 plus 25 by 3 is 11 by 3 plus 7 by 3 minus 5 by 3 is 2 by 3.

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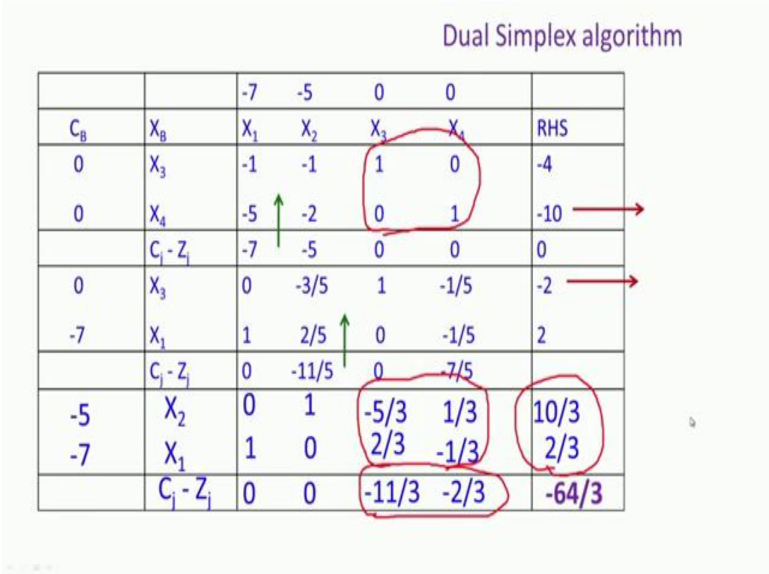

		-7	-5	0	0	
C_B	X_B	X_1	X_2	X_3	X_4	RHS
0	X_3	-1	-1	1	0	-4
0	X_4	-5	-2	0	1	-10
	$C_j - Z_j$	-7	-5	0	0	0
0	X_3	0	-3/5	1	-1/5	-2
-7	X_1	1	2/5	0	-1/5	2
	$C_j - Z_j$	0	-11/5	0	-7/5	
-5	X_2	0	1	-5/3	1/3	10/3
-7	X_1	1	0	2/3	-1/3	2/3
	$C_j - Z_j$	0	0	-11/3	-2/3	-64/3

Now, we see what happens to the table when we did the big m method we realized that the negative of the c j minus z j showed this. So, the negative of the c j minus z j showed the value here, but because we use the big m the identity matrix is here. Now, under that because of the big m we had these values where as we have the minus s of the identity matrix which is here the minus I matrix which is here. So, it is seen under this with the negative values. So, 11 by 3 and 2 by 3 if you go back to the previous slide we will realize that the values are 11 by 3 and 2 by 3. Now, you realize that in this slide we are able to see this 11 by 3 2 to by 3 then we use the big m method. So, this the simplex algorithm based on the big m method. Now, we can also check the b inverse values if you go back to the previous slide b inverse is minus 2 by 3 5 by 3 1 by 3 minus 2 by 3 5 by 3 1 by 3 minus 1 by 3 minus 5 by 3 2 by 3 1 by 3 minus 1 by 3 you can see the b inverse directly under the minus I when we do this there is a minus multiplication which we have to carefully observe and understand that this is the b inverse b inverse is here.

In this case b inverse is actually here. We also show for example, that if we do a dual simplex iteration for the same problem we again see that the primal solution is 2 by 3 10 by 3 the dual solution is 11 by 3 2 by 3 11 by 3 to by 3 you can see the 11 by 3 2 by 3 happening here the 2 by 3 10 x 1 equal to 2 by 3 x 2 equal to 10 by 3. You can also see the b inverse minus 5 by 3 2 by 3 1 by 3 and minus 1 by 3 you can see this. Here you see it with a negative sign because you see minus 2 by 3 5 by 3. Where it was minus 5 by 3 2 by 3 now that happened because x 1 x 2 got interchanged I am treating here x 1 first and then x 2. Whereas in this you realize that x 2 comes first and then x 1 therefore, we got interchanged minus 5 by 3 2 by 3. Where as you realize that it is minus 2 by 3 5 by 3 therefore, what you observe there is a minus 1 multiplication the minus 1 multiplication comes because the dual simplex way here. So, you realize that the original thing is actually a minus matrix, but then we kept the right hand side negative we have multiplying the minus 1. So, the b inverse gets multiplied with minus 1 when we do dual simplex and the negative of the b inverse is shown here which we have to carefully observe whenever we are doing a dual simplex related problems. The b inverse will not be directly visible, but the negative of the b inverse will be visible and we need to understand that aspect.

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Maximize $4X_1 - X_2 + 5X_3$
Subject to
$3X_1 + 4X_2 + 5X_3 \leq 30$
$5X_1 + 8X_2 + 7X_3 \geq 40$
$4X_1 + 6X_2 + 5X_3 \leq 36$
$X_1, X_2, X_3 \geq 0$

$X_B = B^{-1}b$

$Y = C_B B^{-1}$

		4	-1	5	0	0	0		
		X_1	X_2	X_3	X_4	X_5	X_6		
0	X_4	3	4	5	1	0	0	30	
0	X_5	-5	-8	-7	0	1	0	-40	
0	X_6	4	6	5	0	0	1	36	
		4	-1	5	0	0	0		
		0	-7	0	0	0	1	36	
0	X_5	0	-½	-¾	0	1	5/4	5	
0	X_4	0	-½	5/4	1	0	-¾	3	12/5
4	X_1	1	3/2	5/4	0	0	¼	9	36/5
		0	-7	0	0	0	-1	36	

This is the problem that we looked at recently where we had three constraints and two variables I am just showing one of the optimum solutions. Now, if we take if we take this as .This is a greater and equal to constraint. So, if we take the equations as minus 5 x 1 minus 8 x 2 minus 7 x 3 plus x 5 is equal to minus 40 and if x 5 x 4 x 1 are taken. So, for x 5 we would be taking this value 5 minus 7 5 for x 5 will take 0 1 0 for x 4 we will take 1 0 0. For x 1 we will take 3 minus 5 and 4 and if we do the b inverse of that we will get this value we will get b inverse multiplied by 30 40 30 6 would give us straight away would give us this value. So, b inverse into 30 40 30 6 will give us this value the dual will be given by x 4 x 5 x 6 y 1 y 2 y 3 y 3 is equal to 1 and the value will be 36 here x 1 is equal to 9 and the value will be 36. So, you can look at simplex as a method that computes the b inverse for a given basis and then computes the primal solution and the dual solution in a very systematic manner and it moves from one corner point to another, chooses the corresponding basis every corner point differs with one variable coming in one variable going out.

Therefore, it moves from one corner point to an adjacent corner point and keeps traveling this corner points till the optimum solution is found it is always possible to find the solution of the primal as well as a solution of the dual when we do either the simplex table or the dual simplex table or a combination. When we do a dual simplex table the b inverse the negative of it will be shown when we do simplex b inverse will directly be shown. When we do the combination wherever we have that multiplication with the minus one there it will become negative and it gets a little more involved. So, with this we come to an end of our discussion on duality and the dual. So, at this point one question remains we have spend. So, much time in studying in understanding the dual at the same time trying to see where the duel is in a simplex table what is the meaning of the dual and so on. So, we have to ask another question how is this going to help us in our learning of linear programming and operations research and we have to look at that aspect.

Now, simplex is a way by which helped us simplex helped us explain the dual simplex helped us understand that it is solving a primal as well as a dual and every linear programming problem has a dual and we can solve the dual. Another way I have understanding either simplex or the dual simplex is we have a primal feasibility we have a dual feasibility which are linked by complementary slackness. So, if we have an algorithm that maintains primal feasibility and then approaches dual feasibility and gets it then it is optimum. If we have an algorithm that maintains dual feasibility and approaches primal feasibility and gets it then we get the optimum. This principal we need to understand because this principle can be used to create simplex like algorithms. Algorithms that talk about primal feasibility, dual feasibility and complementary slackness and such methods can help us solve different linear programming problems, structure linear programming problems faster and better.

We will now look at two problems in this course we will see what is called the transportation problem and we will see the assignment problem. Both of them are essentially linear programming problems, but then we will look at different algorithms and these algorithms to solve these problems are essentially built around the duality principal. It is built around the principle of having a primal feasible solution and searching for a corresponding dual feasible solution. Or having a dual feasible solution and searching for a corresponding primal feasible solution and if we get that we reach the optimum to these linear programming problems more detailed discussion on transportation problem and assignment problem we will see in the next two modules.