Introduction to Operations Research
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Module - 05
Lecture - 24
Solving LPs with mixed type of constraints

In the last class, we studied the dual simplex algorithm where we solved a minimization problem with all greater than or equal to constraints without using artificial variables. We also said that in the simplex algorithm right through we ensured that the right hand side was greater than or equal to 0 which meant that the primal was feasible and the c j minus z j's where some of them where positive indicating that the dual was infeasible. By entering the positive c j minus z j's we ensured dual feasibility.

So, in this simplex we had primal feasibility and we move towards dual feasibility and then we got the optimum solutions. In the dual simplex we had negative values on the right hand side indicating that the primal could be infeasible, but the dual was always feasible. And then once we decided the leaving variable first and then the entering variable we reached primal feasibility and hence optimum. So, dual simplex ensured that the dual was always feasible and the primal was infeasible, and finally we brought it towards feasibility.

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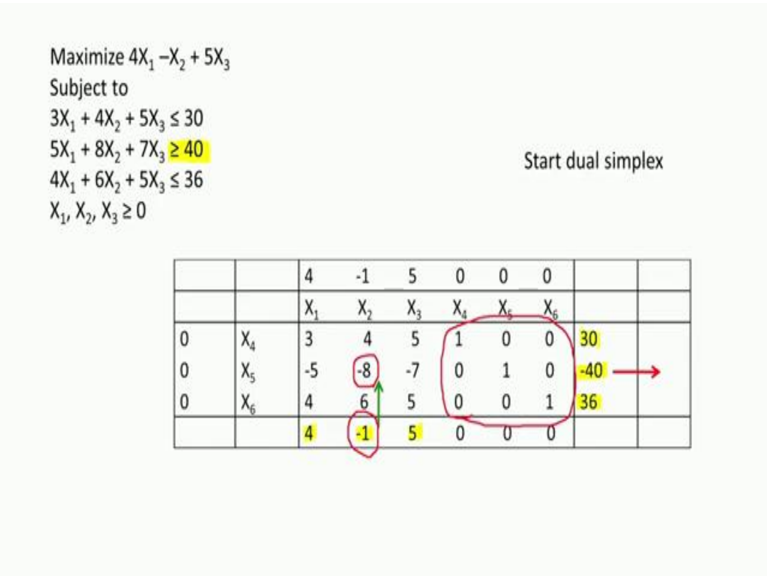

		4	-1	5	0	0	0		
		X_1	X_2	X_3	X_4	X_5	X_6		
0	X_4	3	4	5	1	0	0	30	
0	X_5	-5	-8	-7	0	1	0	-40	
0	X_6	4	6	5	0	0	1	36	
		4	-1	5	0	0	0		

Now, let us look at a problem that is shown here. So, maximization problem with 2 less than equal to constraints, 1 greater than or equal to constraint. The objective function also has a negative term. So, let us first setup the simplex table without artificial variables. Ordinarily the second constraint which had the greater than or equal to would have resulted in a negative slack and therefore, you would have resulted in a artificial variable.

Now, we do not introduce the artificial variable. We retained the negative slack x 5, and then we multiply with the minus 1 so that we get this identity matrix here, using x 4, x 5, x 6, and we can start the simplex table. Now, when we start the simplex table we realize that the right hand side values are positive negative here, and positive indicating that the primal is infeasible. One of the variables has a negative value.

We also calculate the c j minus z j. To begin with, with x 4, x 5 and x 6, there are 3 constraints. So, with x 4, x 5 and x 6 as basic variables, with 0 as the objective function contribution. So, the c j minus z j's are 4, minus 1, 5, 0, 0 and 0. So, to begin with we realize that, we have a situation, if we create a simplex table like this we have a situation where the right hand sides are all, all of them are not greater than or equal to 0; the primal is infeasible.

One of the c j minus z j also has negative; the rest of them have positive. So, this positive indicates that the dual is also infeasible. So, we have a situation, we have a solution where the primal is infeasible as well as the dual is infeasible. Till now in simplex we always maintain primal feasibility; in dual simplex we always maintain dual feasibility. But now we release that we have neither of them satisfied.

Can we still proceed and get a solution to the problem? Or, should we go through the big m method and artificial variable? The answer is we can proceed and do it and we will show how to do it without adding the artificial variable provided the problem has an optimum solution. So, let us try and do this.

So, we could follow what is called the simplex way by first entering a positive c j minus z j and then finding a leaving variable. Or we could do what is called the dual simplex way by first a leaving a variable that has a negative value, so, first identifying the leaving variable and then identifying the entering variable. So, we could do in either of the ways. So, but right now we begin by doing it in the dual simplex way which means we will first identify the leaving variable.

So, the variable x 5 with the minus 40 is the variable that leaves first. So, we have to find out the entering variable. Since we are doing it the dual simplex way we have to find out the coefficient. There is only one negative, so we have to look at only negative c j minus z j values. Problem is a maximization, so we have to look at only the negative c j minus z j values. There is only 1 and it has a negative coefficient in the pivot row.

So, there is only 1 candidate with the ratio of 1 by 8; and variable x 2 is the only candidate and that will enter. So, we can now do the next iteration by replacing variable x 5 with variable x 2. So, we have started it the dual simplex way by saying that x 2 will replace x 5. So, we proceed to the next iteration.

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		4	-1	5	0	0	0		
		X_1	X_2	X_3	X_4	X_5	X_6		
0	X_4	3	4	5	1	0	0	30	
0	X_5	-5	-8	-7	0	1	0	-40	
0	X_6	4	6	5	0	0	1	36	
		4	-1	5	0	0	0		
0	X_4	½	0	3/2	1	½	0	10	20/3
-1	X_2	5/8	1	7/8	0	-1/8	0	5	40/7
0	X_6	¼	0	- ¼	0	¾	1	6	
		37/8	0	47/8	0	-1/8	0		

So, I have shown the next iteration here. So, to begin with we started with this as the leaving variable, and then this has the entering variable. So, let me also use this as an opportunity to explain a problem that has 3 constraints. So far we have solved all our examples with 2 constraints when we did either simplex or dual simplex.

So, this is our pivot element. So, divide every element of the pivot row by the pivot element. So, minus 5 divided by minus 8 will give us 5 by 8. I am highlighting every term that we are doing in this table. Minus 8 divided by minus 8 is 1, minus 7 divided by minus 8 is 7 by 8, 0 divided by minus 8, 1 divided by minus 8 is minus 1 by 8, and 0 minus 40 divided by minus 8 is plus 5. So, we have written the new second row or the pivot row in the next iteration or the new iteration. Now this is called a dual simplex iteration.

And since the pivot element is negative and the right hand side is also negative in the next iteration we will get a positive value here which is a propose of having a negative pivot in a dual simplex iteration. Then we have to do the first row as well as the third row.

Now, to do the first row we look at this element which is 4. Now, we already have created a 1 here which is the new pivot from the pivot element. So, we look at this 4 and then we need a 0 there because we have 3 constraints. Identity matrix is now 1 0 0, 0 1 0, 0 0 1. So, according to x under x 4 we should have 1 0 0, under x 2 we should have 0 1 0, under x 6 we should have 0 0 1. So, we now have a 1, so, we should have a 0, 4 minus 4 times 1 is this 0. So, we can write this first.

So, the first row minus 4 times the new second row is the row operation. So, 3 minus 4 times 5 by 8, 3 minus 20 by 8 which is 4 by 8 which is 1 by 2. 5 minus 4 times 7 by 8 so, 5 minus 28 by 8, 12 by 8, 3 by 2, so, 3 by 2. So, 1 minus 4 times 0 is 1, 0 minus 4 times minus 1 by 8 is plus half, 0 minus 4 time 0 is 0, 30 minus 4 times 520 is 10. So, we have completed this row the first row.

Now, we have to write the third row, the old value, the value is 6. So, 6 minus 6 times 1 will give 0. So, the rule is the new third row is the old third row minus 6 times the new pivot row or the new second row. So, 6 minus 6 times 1 is 0, 4 minus 6 into 5 by 8, so, 4 minus 30 by 8, 32 by 8 minus 30 by 8 is 2 by 8 which is 1 by 4. Off course 6 minus 6 times 1 is 0, 5 minus 6 times minus 1 by 4, 5 minus 6 times 7 by 8. So, 5 minus 42 by 8, 40 by 8 minus 42 by 8 is minus 2 by 8 which becomes minus 1 by 4. 0 minus 6 times 0 is 0. 0 minus 6 times minus 1 by 8 is plus 6 by 8 which is plus 3 by 4. 1 minus 6 times 0 is 1. and 36 minus 6 times 530 is 6. So, this is the next solution. This is how we do the iterations when we have 3 constraints.

Now, the basic variables are x 4, x 2, x 6, in that order you will find the identity matrix in that order 1 0 0, 0 1 0 and 0 0 1. Now we find the c j minus z j values. So, there are 2 0s here. So, we do not have to do the multiplication. The minus 1 is the one that will contribute. So, the first one is 4 minus 5 by 8, 4 minus 4 plus 5 by 8, 4 minus minus 1 into 5 by 8, 4 plus 5 by 8, 32 plus 5, 37 by 8. The basic variables will have 0 as the c j minus z j. So, we need not worry so much about it.

Now we look at the third one. So, 5 minus minus 1 into 7 by 8, 5 plus 7 by 8 which is 47 by 8 and the last one is 0 minus minus 1 into minus 1 by 8 will give us minus 1 by 8. If we want we can calculate the objective function value, but we are not doing it. We could have done it, minus 5 is the value, but I have not shown it because I have one with the negative coefficient, but that does not mean anything; we can calculate the objective function and write it as minus 5; we can do that.

So, now, we realize that we have a iteration at the end of which some of the c j minus z j values are positive. It means we can proceed from the simplex way by entering x 3 which has the largest positive coefficient. We look at the right hand side, now we realize that the right hand side is feasible therefore, we cannot proceed the dual simplex way. So, in this iteration we can proceed only the simplex way and therefore, we proceed in the simplex way by entering variable x 3 which has the largest c j minus z j. So, we do that and we enter variable x 3.

Now to find out the leaving variable we do the simplex way 10 divided by 3 by 2 is 20 by 3, 10 divided by 3 by 2 is 20 by 3, 5 divided by 7 by 8 is 40 by 7; 40 by 7 is less than 6, 20 by 3 is more than 6, so, 40 by 7 is the smaller one. And, 40 by 7 will leave the solution. So, in the next iteration variable x 3 will enter the solution and variable x 2 will leave the solution.

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		4	-1	5	0	0	0		
		X_1	X_2	X_3	X_4	X_5	X_6		
0	X_4	-4/7	-12/7	0	1	5/7	0	10/7	
5	X_3	5/7	8/7	1	0	-1/7	0	40/7	
0	X_6	3/7	2/7	0	0	5/7	1	52/7	
		3/7	-47/7	0	0	5/7	0	200/7	
0	X_5	-4/5	-12/5	0	7/5	1	0	2	
5	X_3	3/5	4/5	1	1/5	0	0	6	10
0	X_6	1	2	0	-1	0	1	6	6
		1	-5	0	-1	0	0	30	
0	X_5	0	-4/5	0	3/5	1	4/5	34/5	34/3
5	X_3	0	-2/5	1	4/5	0	-3/5	12/5	3
4	X_1	1	2	0	-1	0	1	6	
		0	-7	0	$\boxed{0}$	0	-1	36	
0	X_5	0	-½	-¾	0	1	5/4	5	
0	X_4	0	-½	5/4	1	0	-¾	3	12/5
4	X_1	1	3/2	5/4	0	0	¼	9	36/5
		0	-7	$\boxed{0}$	0	0	-1	36	

Now, that is shown here. I have shown a lot of iterations here. So, x 3 has entered here and x 2 had left. So, you see that x 3 has entered the solution and there is no x 2 in this. At the moment I am not going to go through each and every one of the calculations; you can see the simplex table and you can proceed. Now, when we do this the next iteration which is the third iteration of this problem you realize that x 4, x 3 and x 6 are the basic variables.

Now, when we complete the iteration we have a solution x 4 is equal to 10 by 7, x 3 is equal to 40 by 7, and x 6 is equal to 52 by 7 with the objective function value 200 by 7. Now the c j minus z j values are 3 by 7, minus 47 by 7, and 5 by 7, with the x 3, x 4 and x 6 which are the basic variables having 0 as c j minus z j. So, we are interested in this positive c j minus z j, we are interested in this positive c j minus z j.

Now again we realize that in this iteration the primal is feasible. So, 10 by 7, 40 by 7, 52 by 7 is feasible and therefore, we cannot do the dual simplex way; we will do only the simplex way by entering variable x 5 into the solution. Now, to find out the leaving variable 10 by 7 divided by 5 by 7 is 2, 52 by 7 divided by 5 by 7 is 52 by 5. So, 10 by 7 variable, x 4 will leave the solution.

Then we move to the next iteration. So, x 5 will enter, x 4 will leave; you can see this, x 5 has entered, x 4 is not in the solution, now the solution is x 5 equal to 2, x 3 equal to 6, x 6 equal to 6 which is also feasible here. But continuing we realize that there is one c j minus z j which is positive. Therefore, we do the simplex way; we cannot do the dual simplex way. We do the simplex way and then we enter x 1 into the solution. When we enter x 1 into the solution, x 6 will leave the solution with minimum ratio of 6. Therefore, you realize x 1 has now coming to the solution in the place of x 6.

Now, at the end of this iteration you realize that the solution is x 1 equal to 6, x 3 equal to 12 by 5, x 5 equal to 34 by 5 with the objective function value equal to 36. Now the primal is feasible 34 by 5, 12 by 5 and 6. Now, we look at the c j minus z j values. The 3 basic variables which I am highlighting here which I am showing here the 3 basic variables are x 1, x 3 and x 5. So, x 1 has a 0, x 3 has a 0, and x 5 has a 0, under the identity matrix. But if I look at the 3 non basic variables this has a minus 7, this has a 0, this has a minus 1.

So, no positive c j minus z j, algorithm terminates giving the optimum solution which is x 1 equal to 6, x 3 equal to 12 by 5, x 5 equal to 34 by 5, and objective function value equal to 36. The optimum has been reached here at this point; when we have this solution with objective function equal to 36 the optimum has been reached. However, we observe something interesting happening that a non basics c j minus z j which is corresponding to this variable, corresponding to this variable now has a 0 it is not negative it has a 0. Now, can we enter this, is the other question. Now, this phenomenon is called alternate optimum.

So, when the optimum is reached when all c j minus z j less than or equal to 0, I repeat when all c j minus z j less than or equal to 0 and the optimum is reached, and if still the non basic c j minus z j is 0 then it indicates alternate optimum. So, we can enter that and look at the alternate solution. So, if we enter this then this variable will come in and variable x 3 will leave. And if we do another simplex iteration we realize that there is no increase in the objective function because it is an alternate optimum and we find x 1 is equal to 9 with z equal to 36.

Also interesting that we realize that the c j minus z j's are identical; compared to the previous one they will become identical. Once again if we see this optimum is reached, but there is a non basic c j minus z j with the 0 value. There is a non-basic c j minus z j with the 0 value. Non-basic because these values are not identity matrix so, non-basic c j minus z j greater than… So, again if I enter x 3, I will realize that x 4 will leave the solution; and x 3 will come in x 4 will leave I will get this solution; I will get the previous solution will now repeat; the previous solution will now repeat. So, I have to understand that alternate optimum is happening and then I will terminate. So, this is one way to solve a slightly larger problem, a 3 constraint problem. And this problem also helped us understand that alternate optima can happen.

Now, we can do this. To solve this problem we actually started with the dual simplex way. Now, we also said we could have started the simplex way. So, this is the first table. Now, instead of leaving this minus 40 we could have started the simplex way. So, if we started the simplex way then this variable x 3 will come in because this has the largest c j minus z j.

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Maximize $4X_1 - X_2 + 5X_3$

Subject to

$3X_1 + 4X_2 + 5X_3 \le 30$

$5X_1 + 8X_2 + 7X_3 \ge 40$

$4X_1 + 6X_2 + 5X_3 \le 36$

$X_1, X_2, X_3 \ge 0$

Start simplex with X_1 entering

		4	-1	5	0	0	0		
		X_1	X_2	X_3	X_4	X_5	X_6		
0	X_4	3	4	5	1	0	0	30	6
0	X_5	-5	-8	-7	0	1	0	-40	
0	X_6	4	6	5	0	0	1	36	36/5
		4	-1	5	0	0	0		

Now, the ratios are 30 by 5 is 6. Here I do not find the ratio because the pivot element is negative. I would not find the ratio, and 36 by 5, 6 is smaller, so, 6 will go. Now variable x 3 will enter and variable x 4 will leave. And if I do that I come to this and after 4 iterations or 3 iterations I actually get this solution with 36. And once again you realize that there is the alternative optimum and then we can do this and get the solution.

So, if the same problem if we had started the dual simplex way first in this example if we started the dual simplex way first we got the solution, but we did more iterations. When we did the simplex way we also realize that in the next iteration this minus 40 by itself was becoming feasible and therefore, there was no need to do the dual simplex way. So, this also helps us understand that we could do either the simplex way or we could do the dual simplex way, but we will get the final solution with 36 six. Because there is alternate optimum we can do the alternate optimum again and we see the same thing occurring with an alternate solution of 36, and this variable coming in and it will repeat.

Now, interestingly there is a third way to also look at this problem. Now I could have done the simplex way, but interestingly if I have done the simplex way I would have normally gone ahead and entered the variable x 3 because it has the largest c j minus z j. But instead if I choose to enter variable x 1 which does not have the largest c j minus z j if I say any positive c j minus z j can enter x 1 will come in and x 4 will go.

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		4	-1	5	0	0	0		
		X_1	X_2	X_3	X_4	X_5	X_6		
0	X_4	3	4	5	1	0	0	30	
0	X_5	-5	-8	-7	0	1	0	-40	
0	X_6	4	6	5	0	0	1	36	
		4	-1	5	0	0	0		
		0	-7	0	0	0	-1	36	
0	X_5	0	-½	-¾	0	1	5/4	5	
0	X_4	0	-½	5/4	1	0	-¾	3	12/5
4	X_1	1	3/2	5/4	0	0	¼	9	36/5
		0	-7	$\boxed{0}$	0	0	-1	36	

And you realize that in one iteration, we are able to get the optimum solution with z equal to 36; we get this solution with 9. Again alternate optima is indicated by the presence of this 0; and if we do the next iteration we will get the other solution.

So, from this example we have understood a few things: one - if you have a problem with mixed type of constraints if the problem has a solution then we need not add artificial variables, we can do a combination of simplex and dual simplex. There could be iterations where both are possible we can take one of them and proceed; there could be iterations where only one is possible and we proceed in that way.

A general guideline is if both are possible try the simplex thing first and not the dual simplex thing first which means try to first enter a variable with a positive c j minus z j and then proceed. Only if it is absolutely necessary there is no entering and only dual simplex iteration is possible then do the dual simplex.

The other thing we learnt through this example is the presence of alternate optima. So, when we have the optimum reached all c j minus z j's are less than or equal to 0 and then we find that there is a non basic c j minus z j with 0 then it indicates alternate optimum. And we should, we will do that the c j minus z j will repeat and we could take either of these solutions.

In fact, it is also said it is also true that when we have alternate optimum we not only have these 2 solutions we also have infinite alternate optimum solutions. But simplex algorithm being an algorithm that looks at corner points will give us only the corresponding corner points. Some more aspects of solving the dual and understanding the dual solution we will see in the next class.