Introduction to Operations Research
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Module - 05
Lecture - 23
Economic Interpretation of the dual; Dual Simplex algorithm

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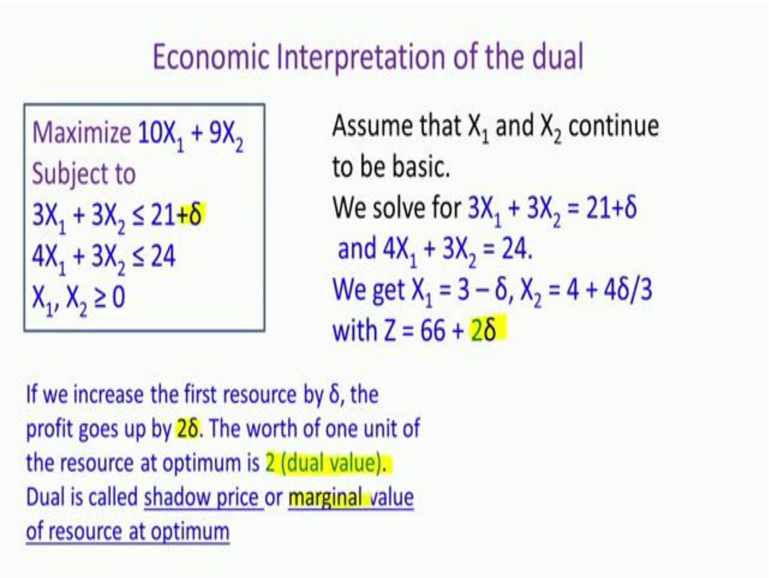

In the last class we addressed the economic interpretation of the dual and we looked at the same example and said if we increase the value of the right hand side of the resource first resource by a small quantity delta and if we solved the problem under the assumption that we make the 2 products x 1 and x 2 then we said that the profit increases or revenue increases by 2 delta. So, if we increase the first resource by a small quantity delta, the profit or revenue goes up by 2 delta, therefore, the worth of one resource the first resource of 1 unit at the optimum is 2 which is the value y 1 equal to 2. So, the dual is called marginal value of the resource at the optimum. It is also called shadow price.

Now, what does that mean? What it means is this. Now, let us assume I have solved this primal and I have also solved the dual and I know that y 1 is equal to 2. Therefore, I will go back and y 1 is equal to 2 and y 2 is equal to 1 which gives me the same 66. It is not the actual price. It is a shadow price because 2 is the worth of the first resource, 1 is the worth of the second resource at the optimum. And together the worth is 66 which is the same as the revenue that is generated by selling these products at 10 and 9 respectively per product to get the same 66.

Now, another way of looking at the problem it is called shadow price because of this. Now let me assume that I have done this little analysis and I have understood that if I increase it by a small quantity delta the revenue will increase by 2 delta. So, for the sake of illustration if 21 becomes 22 then I realize that 66 would become 68. So, I know that this 1 extra unit of this resource can increase the revenue by 2. Therefore, let us assume, I go to the market to buy this extra unit of this resource.

So, when I go to the market and buy this extra unit of the resource, now I am keen about one thing, if I can buy this resource I know that this resource is going to increase the revenue by 2; the extra unit of this resource is going to increase the revenue by 2. So, if this resource is going to, the extra unit is going to cost me more than 2 then I will not buy it because if I buy it for more than 2 my revenue is only going to increase by 2. I will incur a loss; therefore, I will not buy it. If I buy it for less than 2 then I know that I am buying it for less than 2, the revenue is going to be 2 more and then I will make extra profit.

Now, we have to understand the market kind of represents is represented by the dual. And if the market is aware of my dual and has understood that the marginal value of this resource is 2 then the price of this extra unit of this resource is also 2. Therefore, this resource the dual is called marginal value of the resource at the optimum. It is also called the shadow price. It is the notional price in which I can buy the extra unit of that item. Therefore, it is called shadow price or marginal value of the resource at the optimum.

We also need to understand the little thing. Now, is it true that if I know that 1 unit of this resource is going to increase the revenue by 2, can I keep on adding 1 more unit of this resource and keep on increasing the revenue at the same rate? The answer is no. Mathematically putting that question is if delta is 1, so right hand side may become 22, 66 may become 68.

Will that be true for delta equal to 100? The answer is no because as we keep on increasing this delta quantity at some point what will happen is the basis will change. That is the reason when we started we said assume that x 1 and x 2 continue to be basic. At some point as we keep adding these deltas the basis will change and therefore, this proportionality will not happen. And the whole dual, it will result in a new dual solution and then the interpretation has to happen to that new dual solution.

As a matter of exercise one can try instead of keeping this at 21 one can try keeping a large value like 40 and then we will realize that some other solution has emerged and we will not be making both x 1 and x 2. That is the one of the reason why the word at optimum is being used in the dual interpretation. So, it means the worth of the resource at the optimum. So, if I have an optimum solution from that point onwards if I want to add a small delta this is the worth and therefore, this is the price that we have to pay to procure or buy an extra unit of that resource. Now, let us have a further discussion on this by taking another example.

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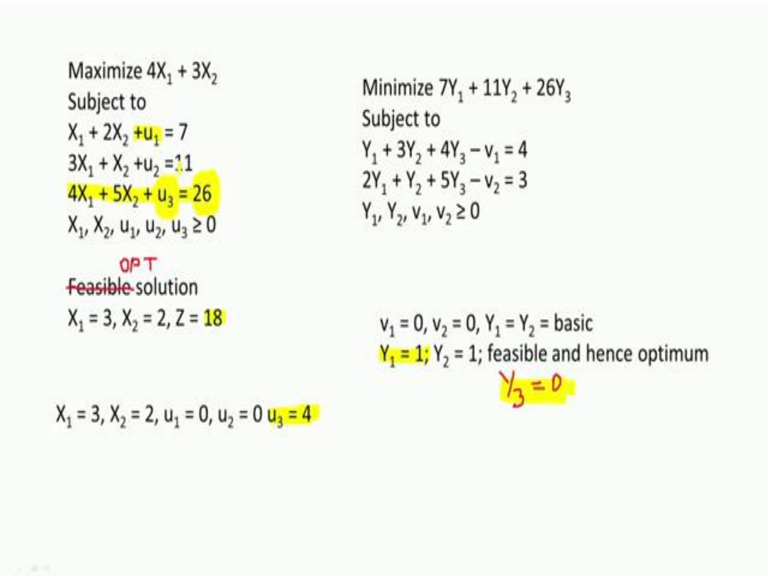

Let us take this example that we have seen just previously to understand the complimentary slackness and the idea of getting the optimum solution of the dual from that of the primal we used this particular example. Now, the optimum solution to this, the dual is written here, the optimum solution is we also know that this z equal to 18 is actually the optimum solution; more than the feasible solution it is optimum solution. So, optimum solution is z is equal to 18 for this. So, u 1, u 2, u 3 are also given. Now, let us apply the complimentary slackness and try to understand the dual solution. Now the dual solution is y 1 equal to 1, y 2 equal to 1; 7 plus 11 is 18. So, each unit worth is 1 feasible and hence optimum.

Now what happens? Now, we realize that if I apply the solution x 1 equal to 3, x 2 equal to 2, now 3 plus 4 is equal to 7; u 1 is equal to 0. So, u 1 equal to 0 necessitates y 1 is in the solution and we get y 1 is equal to 1; u 1 is equal to 0. Now, what is u 1 equal to 0 mean? It means the resource 7 is fully utilized at the optimum. So, when the optimum solution fully utilizes this resource the dual has a value and y 1 is equal to 1. So, an extra unit of the resource has to be bought at a certain price which happens to be 1.

Now, if I take the third constraint, 4 x 1 plus 5 x 2 plus u 3 equal to 26. So, x 1 equal to 3, x 2 equal to 2 means 4 into 3, 12 plus, 5 into 2, 10, 22. So, u 3 is equal to 4 means 4 units of this resource are available; and this resource 26 is not entirely utilized. So, this resource is not fully utilized. So, slack variable is in the solution means the resource is fully not utilized. When the slack variable is in the solution, u 3 is in the solution, y 3 is 0; y 3 becomes 0; y 3 is equal to 0 in this solution.

That is also evident because when is substitute y 1 equal to 1, y 2 equal to 1, y 3 has to be 0. So, when this third resource is not fully utilized which means u 3 is in the solution, the corresponding decision variable y 3 is 0. It means the marginal value of this resource the third resource is 0. So, we have to ask a question, can the marginal value of the third resource be 0? The answer is yes because this resource is not fully utilized.

So, when I am making 3 of the first product and 2 of the second product I have not utilized this resource fully, some resource is already available with me. Therefore, I will not go to the market to pay a price to buy that extra delta. I would rather use it from what I have therefore, the additional resource for this does not, notionally does not have any price associated with it. That is the reason we use marginal value of the resource at the optimum.

If we compare the objective functions 7 into 1 plus 11 into 1 is the same as 18. So, 7 into 1 plus, 11 into 1 plus, 26 into 0 is equal to 18. Now, does it mean that this 26 resources have 0 value per share? Not really. They are worth something. But if I look at it from the optimum solutions point of view because this is not completely utilized I realize that I will not go to the market to buy an extra item at a cost. Therefore, the shadow price is 0, and the marginal value of this resource at the optimum is also 0.

So, when a resource is utilized fully slack is 0, the dual has a value. When the resource is not utilized fully the slack is positive and the dual will not have a marginal value at the optimum. So, this is called economic interpretation of the dual.

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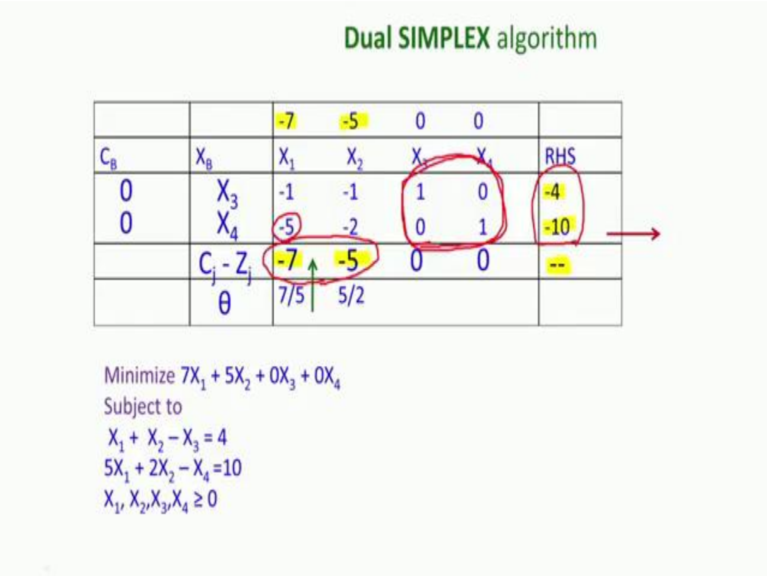

Now let us go to one more aspect of linear programming and try to understand dual simplex algorithm through this example. And then we will also revisit simplex and dual simplex to try to understand the solutions of the primal and the dual, and what is the interpretation. So, let us go to this problem, the minimization problem that we have already seen. Minimize 7 x 1 plus 5 x 2 plus 0 x 3 plus 0 x 4 and so on.

Now x 1 plus x 2 is greater than or equal to 4; 5 x 1 plus 2 x 2 is greater than or equal to 10. So, we have minus x 3 and minus x 4 as 2 negative slack variables. So, when we did simplex we added 2 artificial variables to this and we had a 6 variable problem, we used the big M and we did it. Now let us try to solve this problem in a slightly different manner.

So, this is our first table. So, what has happened? I have multiplied the first constraint with a minus 1, I have multiplied the second constraint with a minus 1 to get minus x 1, minus x 2, plus x 3, plus 0 x 4, is equal to minus 4; minus 5 x 1, minus 2 x, plus 0 x 3, plus 0 x 3, plus 1 x 4 is equal to minus 10. So far when we have done simplex we have never used a negative value on the right hand side.

If we had a negative value on the right hand side we have to multiply with a minus 1, get the right hand side non negative and only then we will proceed. Now we are violating that assumption. We are violating that assumption in the interests of the identity matrix. Now, by doing this I have got an identity matrix for x 3 and x 4. So, if I keep x 3 and x 4 as basic variables I will get an infeasible solution, but the identity matrix I have with me. So, let us see what we do.

So, we now keep x 3 and x 4 as the starting solution because I have the identity matrix. I have x 3 equal to minus 4, x 4 equal to minus 10 which is infeasible. And then the objective function value c b is all 0. Now I do c j minus z j, the values are minus 7, minus 5, 0 and 0, and I do not I put a dash. Now, we have done something which we have never done before. The right hand side values are negative and therefore, this is infeasible.

But if we look at it very carefully what we did in the big m method by actually putting the artificial variables in the solution actually speaking the solution was infeasible that is the reason we did not evaluate the objective function. So, what we are doing now is we are putting it explicitly that it is infeasible by keeping the right hand side values negative. In the big m method we put the artificial variables that took a positive value, but the basic implication was that the solution was still infeasible. Now we are explicitly capturing the infeasibility.

But we also find something interesting. Because it is a minimization problem we convert it to a maximization problem. The objective function coefficients are minus 7 and minus 5. The c j minus z j are minus 7 and minus 5. Now if we understand simplex by looking at the right hand sides and by at looking at c j minus z j, now what happens? The right hand sides are infeasible. The right hand sides are infeasible because I have a negative value on the right hand side.

But surprisingly I am happy with the negative values of c j minus z j here - minus 7, minus 5, 0 and 0. So, if for some reason these 2 were positive then this minus 7, minus 5, 0, 0 will tell me that it is optimum. Now we have also seen that this is this represents a feasibility of the primal, this represents the optimality with respect to the primal, but feasibility with respect to the dual. And we also said the negative of the c j minus z j represents the dual. So, this is now interpreted as the dual is feasible with 7 and 5 the dual is feasible- v 1, v 2; v 1 is equal to 7, v 2 equal to 5, dual is feasible, but the primal is infeasible.

So, what we do? We have to make this primal feasible. So, to make this primal feasible one of the negatives or both the negatives have to go out. So, we first find that variable which is most negative and say that variable goes out. So, here we are doing the opposite we are sending out the variable first. And the variable that has to come in is computed by the ratio which is now done row wise. So, minus 7 divided by minus 5, minus 5 divided by minus 2. So, the smaller number is 7 by 5. So, the variable x 1 comes into the solution theta smaller comes.

The important thing here off course is the pivot has to be negative. So, when we divide we only divide with negative coefficients we do not divide with, for example, we do not do this 0 by 1; negative, negative, so again negative we divide; divide by the negative coefficients so that the pivot element is a negative quantity. Why should the pivot element be negative? If the pivot element is negative you will see in the next iteration this will become positive. So, we show the next iteration this way.

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Dual Simplex algorithm

		-7	-5	0	0	
C_B	X_B	X_1	X_2	X_3	X_4	RHS
0	X_3	-1	-1	1	0	-4
0	X_4	-5	-2	0	1	-10
	$C_j - Z_j$	-7	-5	0	0	0
0	X_3	0	-3/5	1	-1/5	-2
-7	X_1	1	2/5	0	-1/5	2
	$C_j - Z_j$	0	-11/5	0	-7/5	
-5	X_2	0	1	-5/3	1/3	10/3
-7	X_1	1	0	2/3	-1/3	2/3
	$C_j - Z_j$	0	0	-11/3	-2/3	-64/3

So, this is our next iteration. We said this is the pivot element. This pivot element is negative. So, this is our next iteration. So, variable x 1 replaces variable x 4. So, x 3 and x 1 are in the solution; 0 and minus 7. So, again we divide every row of the pivot element by the, every element of the pivot row by the pivot element divided by minus 5 to get these values, and a right hand side value of 2. Now you see that this has become feasible with a non negative value.

Again do the row operations, I need a 0 here. So, minus 1 plus 1 is 0, minus 1 plus 2 by 5 is minus 3 by 5, and so on. And I get minus 4 plus 2 is equal to minus 2. Now I calculate the c j minus z j's which are 0, minus 11 by 5, 0, and minus 7 by 5. Now, once again I realize here that the c j minus z j's are ok, the dual is actually feasible, but the primal is infeasible. Primal is infeasible because of this. The dual is feasible because all these values are negative. Negative of the c j minus z j is the dual.

So, I go back now and push this variable out. This variable has to go out first; x 3 goes out first; I calculate the ratio, minus 11 by 5 by minus 3 by 5 is 11 by 3. Remember, this goes out first; this arrow should come first and then this arrow should come. This goes out first. In this algorithm the leaving variable is first, the entering variable comes later. So, 11 by 5 divided by minus 3 by 5 is 11 by 3; minus 7 by 5 divided by minus 1 by 5 is 7. Once again we divide only with negative values. Smaller one happens to be 11 by 3. So, variable x 2 comes into the solution.

Now, we will do one more iteration, and see what has happens. So, now this is the pivot. So, your pivot is, this is the variable that is coming, this is your pivot. So, x 2 has replaced variable x 3. So, you have x 2, x 1. You have the objective function values. Now divide it by the pivot element 0, 1, minus 5 by 3, 1 by 3 and 10 by 3. Now I need a 0 here. This minus 2 by 5 times 1 is 0. Continue with those operations you will get 2 by 3.

Now, c j minus z j are 0, 0, minus 11 by 3, minus 2 by 3. Now you realize that this solution has the dual feasible. The primal also has become feasible. Therefore, it is optimum. Therefore, it is optimum with minus 64 by 3. Because we converted a minimization problem into a maximization problem the actual value is 64 by 3; x 1 equal to 2 by 3, x 2 equal 10 by 3, 64 by 3. So, this algorithm is called the dual simplex algorithm. It is called the dual simplex algorithm where we do not use artificial variables.

Now, why is it called dual simplex algorithm? If you see carefully in this algorithm, right through the iterations the dual was feasible. The optimality condition was satisfied, c j minus z j's had negatives. So, dual was feasible right through, primal was not feasible. So, we were trying to make the primal feasible. And once the primal became feasible it is optimum.

In a regular simplex you will realize that the primal would always be feasible, the c j minus z j's will have positive values. So, duals were infeasible. And we were making the dual feasible. And once all c j minus z j's were negative we got optimum. In the dual simplex algorithm the dual is always feasible, the primal is infeasible. Once the primal becomes feasible it becomes optimum. So, a simplex and dual simplex ideas put together we can solve linear programming problems extremely effectively.

So, in the class we will take an example to explain how we can use both these principles and what are the implications, and how do we look at the dual solution when we have the dual simplex algorithm, or from the dual simplex algorithm how do we look at the dual solution, we will see those aspects in the next class.