Introduction to Operations Research
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Module – 05
Primal Dual Relatrionships
Lecture - 02
Dual solution from simplex table; economic interpretation of dual

In this class, we continue the discussion on the complimentary slackness and we, we try to explain how the simplex table is able to capture the complementary slackness. In the last class I gave an example where I said, we took two feasible solutions to the primal, one of which happened to be the optimum and I mentioned, that in the, for the same example in the solution that was not optimum, the corresponding dual became infeasible, whereas the one that was optimum, the corresponding dual was feasible from which we said that the solution is optimum.

Now, let us go back to the same example, revisit the simplex table for this problem and try to understand what the simplex algorithm is actually doing.

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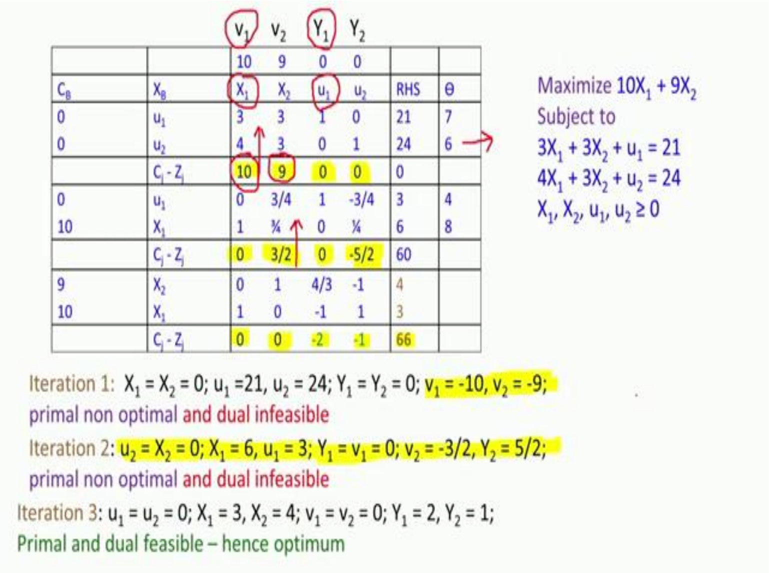

		v_1	v_2	Y_1	Y_2		
		10	9	0	0		
C_B	X_B	X_1	X_2	u_1	u_2	RHS	θ
0	u_1	3	3	1	0	21	7
0	u_2	4	3	0	1	24	6
	$C_j - Z_j$	10	9	0	0	0	
0	u_1	0	3/4	1	-3/4	3	4
10	X_1	1	¾	0	¼	6	8
	$C_j - Z_j$	0	3/2	0	-5/2	60	
9	X_2	0	1	4/3	-1	4	
10	X_1	1	0	-1	1	3	
	$C_j - Z_j$	0	0	-2	-1	66	

So, let us start with this. This is the simplex table. Instead of X 3 and X 4, we have introduced u 1 and u 2. So, you will have X 1, X 2, u 1, u 2 as the variables and then, we start with u 1 and u 2 and then, X 1 comes into the solution replacing u 2 and then, X 2 comes into the solution replacing u 1. And we have the optimum solution with X 1 equal to 3, X 2 equal to 4 and objective function is equal to 66.

Let us now start doing, look at, looking at the simplex table also with the dual in mind. So, if we write the dual to this problem, initially let us assume, that you will be having inequalities, therefore there are two constraints. The dual will have two variables, Y 1 and Y 2. The two constraints give us two slack variables, u 1 and u 2, which are mapped to Y 1 and Y 2 and there are two variables in the primal, which will give rise to two constraints in the dual and they will have minus v 1, minus v 2, where v 1, v 2 are the slack variables. So, the dual will have Y 1, Y 2 and v 1, v 2.

Complimentary slackness also tells us, that there is a relationship between X and v and Y and u. So, X v equal to 0, Y u equal to 0. So, what we do now is, we just map the corresponding dual variable to the primal variable remembering, that the two primal variables X 1, X 2 resulted in two dual constraints, which resulted in two dual slack variables. So, primal decision variable is mapped to the corresponding dual slack variable. X 1 is mapped to v 1 and so on. So, also u 1, which is a primal slack is mapped to Y 1, which is the dual variable.

Now, let us start doing this. The first simplex iteration is with C j minus Z j equal to 10, here C j minus Z j is equal to 9, here 0, 0 and we realize, that this 10 actually entered the solution. The ten entered the solution, we did 7 by 3 and 6 by 4 and then 6 by 4 actually left.

Now, let us try to understand what is actually happening here through, through this. So, the first iteration we had X 1 equal to 0, X 2 equal to 0. From the solution, u 1 is 21, u 2 is 24 implies X 1 and X 2 are non-basic. We can see that. X 1 and X 2 are non-basic, u 1 is 21, u 2 is 24.

Now, let us apply the complimentary slackness conditions from, for this solution. Now, if X 1 is equal to 0, then v 1 is in the solution. X 2 is equal to 0, v 2 is in the solution. u 1 and u 2 are in the solution in the primal, therefore Y 1 and Y 2 are 0. So, one way is to go back to the, write the dual and then try to solve the dual with v 1 and v 2 solving the dual for v 1 and v 2 keeping Y 1, Y 2 is equal to 0. If we do that we will get v 1 is equal to minus 10 and v 2 is equal to minus 9. If we do this, we will get v 1 is equal to minus

10, v 2 is equal to minus 9. If we write the dual and put values, Y 1 and Y 2 equal to 0, but let us look at what the simplex table says.

Now, the simplex table shows a 10 here and a 9 here. Now, the negative of the C j minus Z j seem to show the corresponding dual solution. Now, if we realize this, if you see u 1 is in the solution in the primal. So, Y 1 has to be equal to 0. We have now mapped Y 1 alongside u 1, go down, here you get a 0 here, you get a 0 here, this is the Y 1 solution. Similarly, u 2 is in the solution in the primal by complimentary slackness, Y 2, u 2 is 0. Therefore, Y 2 has to be 0. Now, Y 2 is mapped to u 2. So, this is the value for Y 2; y, Y 2 is 0. That is precisely why the C j minus Z j for every basic variable has to be 0 because when that variable is basic, the corresponding variable in the dual will be non-basic and it will take the value 0.

Now, we realize X 1 is equal to 0, non-basic in the primal. Its corresponding dual variable v 1 is actually basic and has a value minus 10. So, negative of the C j minus Z j is able to show the dual solution. Similarly, X 2 is non-basic in this solution. X 2 is 0. So, X 2, v 2 is 0. We realize v 2 is minus 9, which is this, which we would have obtained otherwise by applying complimentary slackness. So, right here in the very first iteration we are actually solving a feasible solution to the primal with u 1 equal to 21, u 2 equal to 24. This is not only infeasible, but it is basic feasible and through the C j minus Z j values we are actually evaluating the dual solution.

Now, the dual solution is Y 1 equal to 0, Y 2 equal to 0, v 1 is equal to minus 10, v 2 is equal to minus 9. Now, v 1, v 2 are infeasible. So, we look at the C j minus Z j and enter that variable with a positive C j minus Z j.

Now, let us say we do one more simplex iteration after which we get X 1 equal to 6, u 1 equal to 3 with Z equal to 60. Now, let us see what happens to that. Now, our 2nd iteration is X 1 equal to 6, X 1 equal to 6, u 1 equal to 3. So, when I substitute X 1 equal to 6, u 1 equal to 3, X 1 equal to 6 here will give u 2 equal to 0. X 1 equal to 6 and u 1 equal to 3 will also give me X 2 equal to 0 and it is fairly obvious, that u 1, X 1 are in the solution, u 2, X 2 are 0. They are non-basic with 0. So, this is the primal feasible solution, primal basic feasible solution.

Now, let us apply complimentary slackness. When u 1 is in the solution, Y 1 has to be 0. So, Y 1 is mapped to u 1, you will realize Y 1 is 0. The C j minus Z j of a basic variable is 0. X 1 is in the solution, therefore v 1 has to be 0. Therefore, you see, under X 1, v 1 is mapped, v 2 is v 1 is 0. Now, if we apply the complimentary slackness with Y 1 and v 1 to 0 and we write the dual and we solve for Y 2 and u 2, we will get Y 2 is equal to 5 by 2, v 2 is equal to minus 3 by 2. So, go back and see, that 5 by 2 is shown as minus 5 by 2 here, as I said, negative of the C j minus Z j. And this 3 by 2 is shown as the minus 3 by 2 is shown as 3 by 2, which is the negative value.

So, again what we do in this iteration is, we are looking at another feasible solution to the primal with X 1 equal to 6, u 1 equal to 3 and we are evaluating the corresponding dual solution. The corresponding dual solution is still infeasible because Y 2 is positive, but v 2 is negative, negative of the C j minus Z j. So, v 2 is negative, v 2 is negative, therefore it is infeasible and therefore, we look at C j minus Z j. Again, this variable comes into the solution. We now realize, that this variable is coming into the solution with a positive C j minus Z j and then we do one more iteration and finally, get X 1 equal to 3, X 2 equal to 4. When X 1 equal to 3, X 2 equal to 4, automatically u 1, u 2 are 0 and you realize, that u 1, u 2 are 0.

So, X 1 equal to 3 implies v 1 is equal to 0. So, v 1 is equal to 0. X 2 equal to 4 implies v 2 is equal to 0. And when we put v 1 equal to 0, v 2 equal to 0 into the dual and we actually solve, we will get Y 1 equal to 2, Y 2 equal to 1, negative of the C j minus Z j. So, Y 1 equal to 2, Y 2 equal to 1 is actually feasible to the dual and therefore, it is optimum with 66. Now, you realize that is shown here. So, in the iteration three, you realize Y 1 equal to 2, Y 2 equal to 1, it is optimum. So, negative of the C j minus Z j is actually the value of the dual and under u 1, u 2 we will be able to see the value of Y 1 Y 2 with a negative sign. So, 2 and 1 is the solution.

So, what does simplex algorithm do? There are two ways of looking at the simplex algorithm. The first way is, simplex looks at a basic feasible solution to the primal, evaluates C j minus Z j as a way to check whether the objective function can be further increased. So, with 10, 9, 0, 0, it can be further increased with 0, 3 by 2, 0, minus 5 by 2. it can be further increased, but with 0, 0, minus 2, minus 1. It cannot be further increased, therefore the optimum is reached. That is one way of looking at the simplex algorithm.

The other way of looking at the simplex algorithm is, I evaluate a basic feasible solution to the primal through the C j minus Z j and the negative values of the C j minus Z j, we look at minus 10 and minus 9 as the dual solution. Dual is infeasible. So, when the dual is infeasible and you can see, that the dual slacks are 10 and 9, which means, there is a gap in the dual solution.

For example, the first solution would be from the dual, it will be 3Y 1 plus 4Y 2 greater than or equal to 10. So, Y 1, Y 2 are 0. Therefore, v 1 becomes minus 10. So, there is a gap between the left hand side and the right hand side of the dual constraint. When a slack variable takes a negative value, v 1 takes a negative value, v has to be greater than or equal to 0. But if v 1 takes a negative value, one of the constraints is violated and there is a gap. Now, here both the constraints are violated.

Now, we try to make the dual feasible by bridging the gap. Now, this is the variable that has the most negative value. v 2 is minus 10 where the gap is large and the violation is large, which has resulted in a positive C j minus Z j of 10. So, entering a positive C j minus Z j is the same as making an infeasible constraint feasible, making an infeasible dual constraint feasible. So, X 1 comes into the solution so that this v 1 that is negative will try to become 0 in the next iteration. So, primal non-optimality, one way of saying a positive C j minus Z j indicates non-optimality of the primal. So, most positive put it in.

The other way is to say, a positive C j minus Z j indicates infeasibility of the dual because the negative of the C j minus Z j is the dual. So, here there are positive C j minus Z j's indicating non-optimality. Here, there are positive C j minus Z j's for the X 2 indicating non-optimality. Here there is no positive C j minus Z j, therefore optimum.

Now, same thing is told differently. Here, the dual is infeasible. So, bridge the gap by pushing the one that has the largest gap. Here, the dual is infeasible, bridge the gap by pushing v 2 into the solution so that v 2 from a negative value will become 0. Now, here the dual is feasible, therefore it is optimum. So, simplex does the same thing. In a, in a, in a singled solution it actually solves the primal as well as the dual to a given linear programming problem and that can best understood if we look at complimentary slackness conditions.

Now, let us apply, let us look at some more aspects of the primal and the dual.

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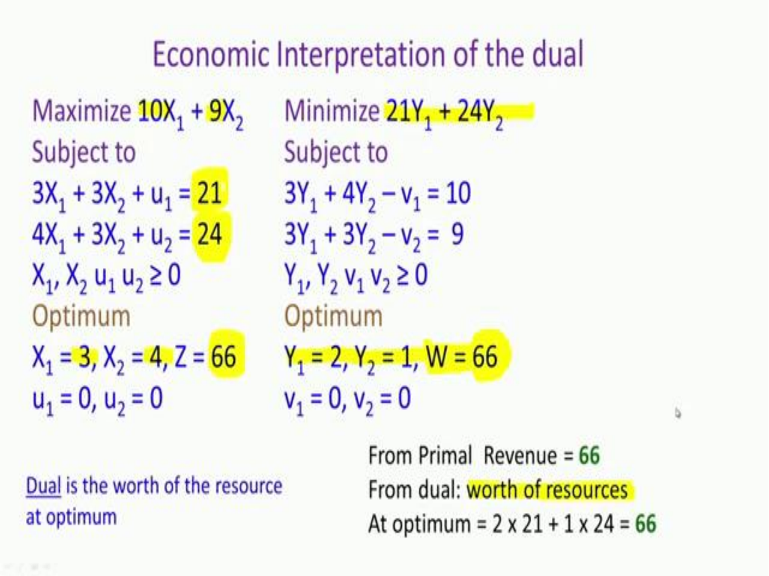

Now, we start something called the economic interpretation of the dual. Now, what is the meaning of the dual? Does it have a meaning? How is it related to the primal? Is there an economic angle? Is there a money angle, a monetary angle to looking at the dual? So, let us start the economic interpretation of the dual in this class and continue as we move along. So, again we use the same example to illustrate so that we can have a better understanding of linear programming because we have seen this example so many times already and we will continue to see it for some more time. We can understand a lot more things by keeping the same example.

So, we know that primal, we know the dual already, we know the optimum solution X 1 equal to 3, X 2 equal to 4, Z equal to 66, u 1 equal to 0, u 2 equal to 0, y 1 equal to 2, y 2 equal to 1, etcetera. Now, what, what is the first thing we understand? Now, from the primal, the revenue is 66; the revenue is 66. If you look at it, this is the problem with which we started. So, we made two products, there are two resources, these are the revenues associated with selling the product 10 and 9 and finally, we solved to say, make three of the first product and make four of the second product, two types of sweets. So, make three of the first product and four of the second product to get a revenue of 66.

Now, what does the dual tell me? The dual says, y, Y 1 is equal to 2, Y 2 equal to 1 with W equal to 66. Now, this W equal to 66 comes from 2 into 21, 42 plus 24 is 66, is one way of looking at it. The other way is 2 into this 21 plus 1 into 24 is 66. The right hand side of the primal is the objective function of the dual. So, before explaining what Y 1 and Y 2 are, we also understand, that if Y 1 and Y 2 represent something 21 and 24 are the resources with which we started the problem. And if Y 1 and Y 2 represent something 21Y 1 plus 24Y 2 is equal to 66 which is the revenue that we got by selling the products and making this 66.

So, the first understanding is, if 21 into something plus 24 into something is also equal to 66, then this Y 1 and Y 2 have something to do with 21 and 24. It has to do with the resources and we now say, that the work of the resources is 66 at the optimum. It is also to say, that if I have 21 units of the first resource and 24 units of the second resource as resources and then I transform them into products and make three of the first product and four of the second product and sell the products to realize the revenue of 66. It means, the actual worth of the two resources, which is 21 and 24 units of the resource, the worth of these two resources at the optimum is the same 66 that we have. So, the first interpretation is the dual. Variable values are worth of the resources at the optimum.

We will qualify it a little more, but they are the worth of the resources at the optimum. It is also good to say worth of the resources at the optimum. So, first economic interpretation is Y 1 and Y 2, which are the dual variables, are the worth of the resources at the optimum.

Now, let us understand a few more things about…

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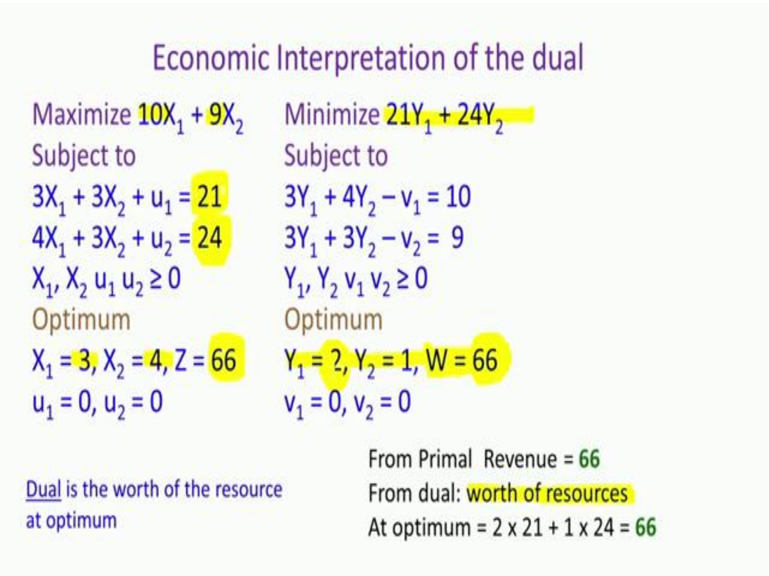

Once again let us go back to the same example. The original problem was 3X 1 plus 3X 2 is less than equal to 21 and 4X 1 plus 3X 2 less than equal to 24. Now, let us assume, that we are going to solve the same linear programming problem. Instead of having 21 units of the resource, we are going to assume, that we will have 21 plus delta, where delta is a small positive quantity. You can imagine 21.1 or something like that, a small quantity, what if, what the resource increases a little bit, 21 plus delta.

We are also going to make an assumption that because of this increase of delta we will still continue to make both the products, which means, X 1 and X 2 that were in the solution. We know the solution X 1 and X 2 were in the solution. We will, we will assume, that we will continue to make both the products and under that assumption what will happen is, if we, we will assume, that X 1 and X 2 will continue to be basic. We will continue to make the two products, then it is enough to solve for 3X 1 plus 3X 2 is equal to 21 plus delta and 4X 1 plus 3X 2 is equal to 24. Assume, that delta is known. So, when 3X 1 plus 3X 2 is equal to 21 plus delta and 4X 1 plus 3X 2 is equal to 24. If we actually solve, we will get X 1 is equal to 3 minus delta, X 2 is equal to 4 plus 4 delta by 3 and when we substitute, the objective function will be 66 plus 2 delta.

So, what does it tell me? It tells me, that if I have a small delta extra in the first resource, I can now convert this delta into a product and finally, sell these products and the revenue instead of 66 will become 66 plus 2 delta. So, that is what this tells me. Therefore, if we increase the first resource by a small delta, the profit or revenue goes up by 2 delta. The worth of one unit of the resource at optimum is 2, which was the dual solution Y 1 is equal to 2. If you go back to the previous slide we know, that Y 1 is equal to 2. You see here, that Y 1 is equal to 2; Y 1 is equal to 2. So, this is actually the worth of that resource. Therefore, you realize 2 into 21 plus 1 into 24 was 66. So, Y 1 is equal to 2.

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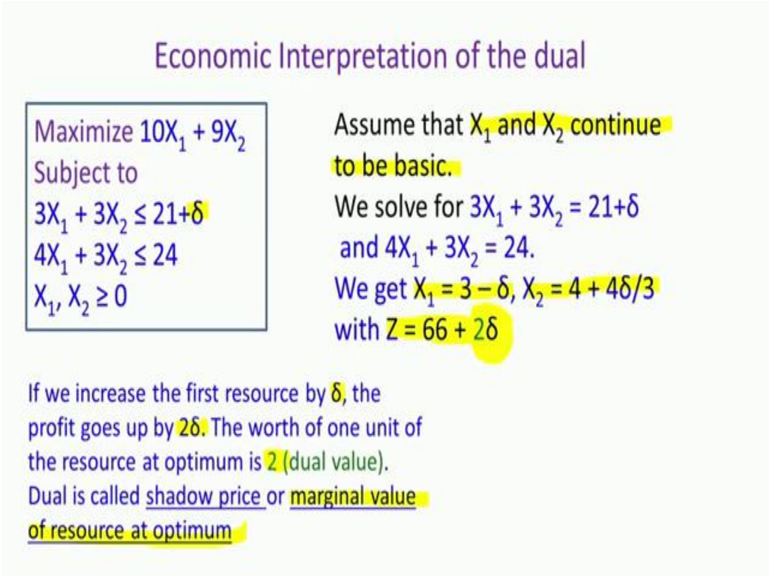

So, we realize, that if we increase the first resource by delta, which is shown here 21 by delta, the thing goes up by 2 delta 66 plus 2 delta. So, therefore this 2, which happens to be the value of Y 1 is the marginal value of the resource at optimum. Now, at the optimum, if, at the optimum if I increase the first resource by a small delta, then the profit goes up by 2 delta. Therefore, the worth of this resource, the marginal value of this resource, the worth of this resource at the optimum is indeed 2.

We will see some more aspects of the economic interpretation of the dual and then move on to the dual simplex algorithm in the next class.