Introduction to Operations Research
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Module - 05
Primal Dual Relationships
Lecture - 01
Dual solution using complimentary slackness

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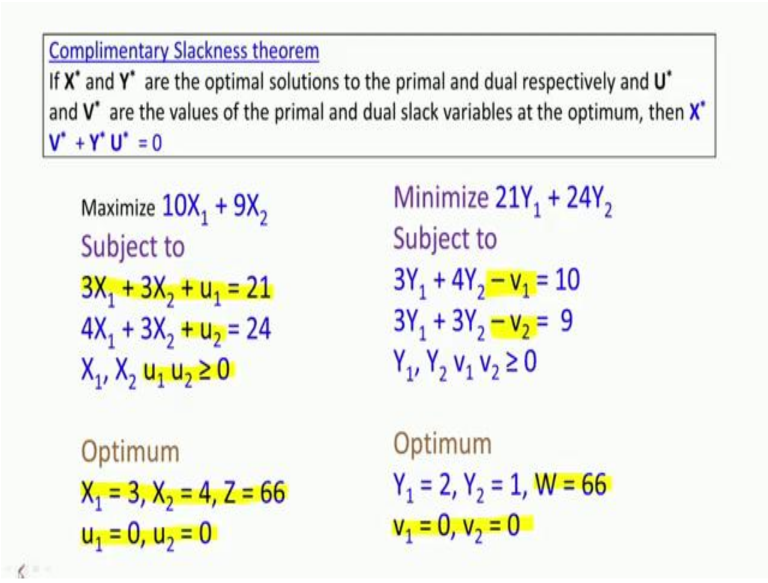

In this class, we start with the complimentary slackness theorem, which says, if X star and Y star are optimal solutions to primal and dual and U star and V star are the values of the primal and dual slack variables at the optimum, then X star into V star plus Y star into U star is equal to 0.

We also mentioned in the previous class that since all X's, X 1, X 2, Y 1, Y 2, U 1, U 2 and V 1, V 2 are greater than or equal to 0, X V plus Y U or X star V star plus Y star U star equal to 0 will become X 1 V 1 equal to 0, X 2 V 2 equal to 0, Y 1 U 1 equal to 0 and Y 2 U 2 equal to 0.

We also know, that U 1 and U 2 are the slack variables corresponding to the primal and V 1 and V 2 are the slack variables corresponding to the dual, and there is a relationship between X and V because X is the number of variables in the primal, which is equal to the number of constraints in the dual and therefore, the number of variables in the primal will be equal to the number of slack variables in the dual. Similarly, there is a connection between Y and U because U is the number of slack variables in the primal, which is equal to the number of constraints in the primal and since the number of constraints in the primal is equal to the number of variables in the dual, there is a connection between Y and U.

We also verified the complimentary slackness by considering this example, which maximizes 10 X 1 plus 9 X 2 subject to 3 X 1 plus 3 X 2 plus u 1 equal to 21. 4 X 1 plus 3 X 2 plus u 2 equal to 24. The dual is shown here; the dual is shown here. And we also said, that in the optimum we said, X 1 is equal to 3. So, complimentary slackness says, X 1 equal to 3 V 1 should be equal to 0.

From the optimum solution to the dual we realize, that Y 1 equal to 2, Y 2 equal to 1. When we substitute in the first constraint, 3 into Y 1 plus 4 into Y 2 is equal to 10, therefore v 1 is equal to 0. So, we realize, that X 1, V 1 is equal to 0. Similarly, from the optimum solution to the primal, X 2 is 4.

Now, from the dual, when we substitute Y 1 equal to 2 and Y 2 equal to 1 in the second constraint, we get 6 plus 3 equal to 9. Therefore, v 2 is equal to 0. So, now we realize, that X 2 is 4, v 2 is 0. So, X 2, V 2 is 0. Similarly, from the primal, from the first constraint, when we substitute X 1 equal to 3, X 2 equal to 4, we get 3 into 3, 9 plus 3 into 4, 12. 9 plus 12 is equal to 21. Therefore, u 1 is equal to 0.

Now, from the optimum solution to the dual, Y 1 is equal to 2, therefore Y 1, u 1 is equal to 0. Similarly, from the second constraint of the primal, when we substitute X 1 equal to 3 and X 2 equal to 4, we get 4 X 1 plus 3 X 2 is equal to 24. Therefore, u 2 is equal to 0 and we realize from the dual, Y 2 is equal to 1. So, Y 2 into u 2 is equal to 0. Therefore, the complimentary slackness is satisfied.

We also observe one more thing. We can also say, that for example, we take the primal and X 1 is equal to 3, X 2 equal to 4 is the solution. So, when we substitute 3 and 4, the original constraint was 3 X 1 plus 3 X 2 less than or equal to 21. So, when we substitute X 1 equal to 3 and X 2 equal to 4, we will get LHS equal to RHS. Therefore, u 1 will be equal to 0 and when u 1 is equal to 0, we realize Y 1 is in the solution.

So, another way of saying is, if the constraint is satisfied, the original inequality is satisfied as an equation, then the corresponding slack will be 0 and therefore, the corresponding dual variable will be in the solution. Similarly, 4 X 1 plus 3 X 2 itself becomes 24. So, when the optimum solution to the primal satisfies one of the inequalities as an equation, then it means, that the corresponding slack variable is 0. And when the corresponding slack variable is 0, the corresponding basic variable in the corresponding decision variable or variable in the dual will become basic. We can observe that from this example in addition to understanding, that the complimentary slackness conditions have been satisfied

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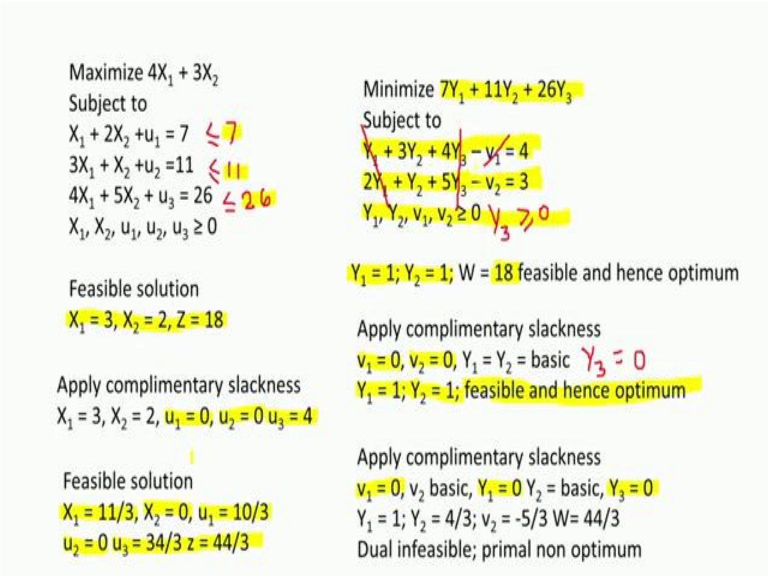

We will now move on to another example to understand little more aspects of the complimentary slackness theorem and the usefulness of complimentary slackness theorem.

So, we look at another linear programming problem this time with two variables and three constraints. So, maximize 4 X 1 plus 3 X 2 subject to X 1 plus 2 X 2 less than or equal to 7; X 1 plus 2 X 2 less than or equal to 7. And when we add the slack variable u 1, it will become X 1 plus 2 X 2 plus u 1 equal to 7. Similarly, 3 X 1 plus X 2 less than or equal to 11. So, we add another slack variable, which is 3 X 1 plus X 2 plus u 2 equal to 11. Similarly, 4 X 1 plus 5 X 2 is less than or equal to 26 and when we add a slack variable, it becomes plus u 3 equal to 26. X 1, X 2, u 1, u 2, 3 greater than or equal to 0. Obviously, the contributions of u 1, u 2 and u 3 to the objective function is also 0.

We now write the dual of this problem. Now, the primal has two variables excluding the slack variables. So, the dual will have two constraints, the primal has three constraints and the dual will have three variables excluding the slack variable. So, the dual is now written and then the slack variables are added to the dual. So, the dual is, initially we will write the dual as a minimization problem. The primal has two variables without the slack variables. So, the dual will have two constraints and the primal has three constraints therefore, the dual will have three variables.

So, we will first write the dual with Y 1, Y 2, Y 3 and later we have added the slack variable. So, when we write it with Y 1, Y 2, Y 3, the dual will be to minimize 7 Y 1 plus 11 Y 2 plus 26 Y 3. 7 Y 1 plus 11 Y 2 plus 26 Y 3, which you find here, Y 3 subject to Y 1 plus 3 Y 2 plus 4 Y 3 greater than or equal to 4. So, Y 1 plus 3 Y 2 plus 4 Y 3 greater than or equal to 4 and with the addition of a negative slack becomes minus v 1 is equal to 4.

The second constraint will be 2 Y 1 plus Y 2 plus 5 Y 3 greater than or equal to 3. So, 2 Y 1 plus Y 2 plus 5 Y 3 greater than or equal to 3 and with the addition of the negative slack minus v 2 is equal to 3. Now, Y 1, Y 2, v 1, v 2, v 3 greater than or equal to 0. V 3 is also greater than, I am sorry, Y 1, Y 2, Y 3, there are three variables. Y 1, Y 2, Y 3, v 1, v 2 are greater than or equal to 0. Y 3 is also greater than or equal to 0 and that has to be included. So, we have all the five variables greater than or equal to 0. So, this is how we will write the dual when we have unequal number of variables and constraints in the primal.

And we also have to understand, that when we first write the dual, we safely write the, we treat the primal with the inequalities and then write the dual and then add sufficient slack variables to convert the dual constraints to equations. Now, what do we do with this primal and dual? So, let us do a few things.

Now, let us first look at the primal and observe, that we have a solution X 1 equal to 3, X 2 equal to 2 with objective function equal to 18, is feasible to the primal. So, let us verify that. So, X 1 equal to 3, X 2 equal to 2 is 3 plus 4 is equal to 7, feasible; 3 into 3, 9 plus 2 equal to 11, feasible; 3 into 4, 12 plus 2 into 5, 10. 22 is less than equal to 26 is also feasible. Now, because plus u 3 equal to 26, we will have u 3 equal to 4. So, this solution is feasible with objective function value equal to 18. 3 into 4, 12 plus 2 into 3, 6 is equal to 18. Now, what do we do?

Now, let us look at the dual independently and then, let us look at a solution, Y 1 equal 1, Y 2 equal to 1 as another solution. So, if we look at this solution, Y 1 equal to 1, Y 2 equal to 1 in isolation, we will go back and see, that Y 1 plus 3, Y 2 is 4, Y 3 is 0. So, 4, v 1 is also 0. Therefore, this constraint is satisfied. Similarly, 2 Y 1 plus Y 2 Y 3 is 0, v 2 is zero. So, equal to three this constraint is also satisfied. Y 1, Y 2, Y 3, v 1, v 2 greater than or equal to 0. Y 3, v 1 and v 2 are 0, therefore all the constraints including the non-negativity are satisfied. So, this is independently another feasible solution to the dual.

Now, what is the value of the objective function for this? Y 1 equal to 1, Y 2 equal to 1 would give us 7 plus 11, which is equal to 18. So, I have another feasible solution to the dual. I have a feasible solution to the primal, I have a feasible solution to the dual. Therefore, based on the optimality criterion theorem I can now say, that both these are optimum to primal and dual respectively.

So, that is one way to understand primal dual relationship. Of course, this does not have much to do with the complementary slackness. We will look at that again. We will see, we will apply complimentary slackness and try to understand. The first thing is, if we are able to get a feasible solution to the primal, we are able to get another feasible solution to the dual and if they happen to have the same value of the objective function, then based on the optimality criterion theorem, both are optimum to primal and dual respectively. But then, we will ask another question. It is not also that easy to get two feasible solutions independently, one each to the primal and dual having the same value of the objective function. So, now, we will apply complimentary slackness and try to understand a few things.

So, let us go back to the same feasible solution here with X 1 equal to 3, X 2 equal to 2 and Z equal to 18. Let us now not think of this solution Y 1 equal to 1, Y 2 equal to 1. Let us not think of this solution. So, now let us apply complimentary slackness to the, to this solution. So, first let us define all the variables so that we can apply the complimentary slackness.

So, when we apply the complimentary slackness, X 1 equal to 3, X 2 equals to 2, we go back and substitute in the three equations now. So, 3 plus 4, 7 plus u 1 is equal to 7, therefore u 1 is equal to 0. 3 into 3, 9 plus 2 is equal to 11, therefore u 2 is equal to 0. Now, 3 into 4, 12 plus 5 into 2, 10 is 22, therefore u 3 is equal to 4. So, this is the complete primal solution, which also includes the non-basic variables as well as the slack variables.

If we had actually solved this by the simplex algorithm, we would have got X 1 equal to 3, X 2 equal to 2 and u 3 equal to 4. So, we now have the solution to the primal and we now apply the complimentary slackness conditions and see what happens to the dual.

Now, X 1 is equal to 3, X 1 is in the solution, X 1, v 1 should be equal to 0, therefore v 1 should be 0. X 2 is in the solution, X 2 equal to 2, X 2, v 2 should be equal to 0, therefore v 2 is equal to 0. Now, u 1 equal to 0, y 1 u 1 is 0, so it is possible, that Y 1 is in the solution. u 2 is 0, therefore it is possible, that Y 2 is in the solution. u 3 is equal to 4 implies Y 3 has to be equal to 0 because u 3 y 3 is 0. So, now, out of these five variables in the dual, Y 1, Y 2, Y 3, v 1, v 2, we have now understood, that v 1, v 2 and Y 3 are 0, which means, Y 1 and Y 2 have to be in the solution because the dual has two constraints. So, let us go back and see what happens when we apply the complimentary slackness condition.

So, what I mentioned happens. Because X 1 is equal to 3, we will have v 1 is equal to 0. Because X 2 is equal to 2, we will have v 2 is equal to 0. Because u 3 equal to 4, we will also have Y 3 is equal to 0. So, let me write that here. So, I will write, Y 3 is equal to 0 and because I have u 1 equal to 0 and u 2 equal to 0, so they have to Y 1 and Y 2 can be in the solution.

But since there are two equations, they are in the solution and therefore, it is now enough for me to solve this dual only for Y 1 and Y 2, which means, I can leave out Y 3, v 1 and v 2. It is enough to solve only for Y 1 and Y 2. So, when I put Y 1 plus 3Y 2 equal to 4, 2Y 1 plus Y 2 equal to 3, I will get a solution. I just solve only for these two Y 1 plus 3, Y 2 equal to 4, 2Y 1 plus Y 2 equal to 3.

So, when I solve for this, when I multiply the first equation by 2, I will get 2Y 1 plus 6Y 2 is equal to 8 and I subtract 5Y 2 is equal to 5. I will get Y 2 equal to 1 and I will substitute to get Y 1 equal to 1. So, I get Y 1 equal to 1, Y 2 equal to 1. Now, this solution Y 1 equal to 1, Y 2 equal to 1, v 1, v 2, Y 3 equal to 0 is feasible and hence, optimum.

So, now I am not applying optimality criterion theorem. I have not evaluated the objective function of the dual. Now, I say, that I have a feasible solution to the primal, I apply complimentary slackness and solve, what, what I call as a corresponding dual solution. And if the corresponding dual solution is feasible, then it is optimum. It will also happen, that the objective function value will be equal and will be 18, but I am not showing optimality based on the optimality criterion theorem, but I am showing optimality based on complimentary slackness. So, I have a feasible solution with z equal to 18. I apply the complimentary slackness and evaluate a corresponding dual solution that happens to be feasible and therefore, it is optimum.

Now, let me look at another instance. Now, let me go back to the same problem, to the primal and let me take a solution, X 1 is equal to 11 by 3. So, let me use X 1 is equal to 11 by 3. From the second constraint I am writing, X 1 is equal to 11 by 3. So, when I have X 1 is equal 11 by three automatically; X 2 is equal to 0 and u 2 is equal to 0. You can see, that X 1 is equal to 11 by 3, X 2 will be 0, u 2 will be 0. Now, I go back and substitute when X 1 is equal to 11 by 3, u 1 will be 7 minus 11 by 3, which is 10 by 3. When I substitute here, 11 by 33 into 4 is 44 by 3. So, 25 minus 44 by 3 is 34 by 3. 78 minus 44 is 34 by 3 and then objective function value is 44 by 3, 11 by 3 into 4.

Now, here I have a feasible solution to the primal. Now, let me apply complimentary slackness and see what happens. Now, when I apply complimentary slackness, X 1 is in the solution. So, X 1 is in the solution, v 1 is equal to 0; u 1 is in the solution, Y 1 is equal to 0; u 3 is in the solution, Y 3 is equal to 0. So, that leaves me with v 2 and Y 2 in the solution. So, v 2 and Y 2 are in the solution.

So, I will go back and say, v 2 and Y 2 are in the solution. So, Y 1 is not in the solution, Y 2 is in the solution, Y 3 is not in the solution, v 1 is not in the solution. So, this will give me Y 2 is equal to 4 by 3. From here, Y 2, 3Y 2 is equal to 4. So, Y 2 is equal to 4 by 3, then I go back and substitute. So, v 2 is in the solution. So, here I will have v 2 is equal to minus 5 by 3, Y 1 will be 0. So, here Y 1 will be 0. So, Y 1 is 0 because u 1 is in the solution. So, Y 1 is 0 and I will have v 2 is equal to minus 5 by 3 in the solution and I have the same value of objective function, which is 44 by 3. Now, I look at this dual solution and I realize, that this dual solution is infeasible because y, v 2 is equal to minus 5 by 3.

So, when I have a feasible solution to the primal that is not optimum and if I apply complimentary slackness and find a corresponding dual solution, I observe, that the corresponding dual solution is infeasible and therefore, it is not. Only when the primal is optimum, I, if, and if I apply complimentary slackness, the corresponding dual after applying complimentary slackness will be feasible and both will be optimum to primal and dual respectively.

Some more aspects of complementary slackness we will see in the next class.