Introduction to Operations Research

Prof. G. Srinivasan
Department of Management Studies
Indian Institute of Technology, Madras

Module - 04
Dual
Lecture - 05
Complimentary Slackness Theorem

In the last class, we saw the weak Duality theorem and the Optimality criterion theorem. The weak duality theorem tried to relate feasible solutions to the primal and the dual. And the weak duality theorem says that, for maximization primal, every feasible solution to the dual will have an objective function value greater than or equal to that of every feasible solution to the primal.

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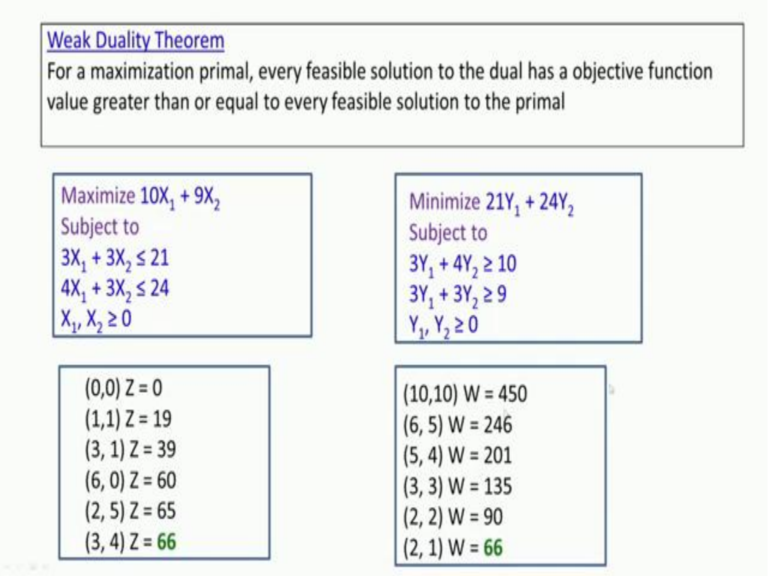

So, here we saw the weak duality theorem and understood that every feasible solution to the dual will have an objective function value greater than that of every feasible solution to the maximization problem. So, we tried to relate the feasible solutions.

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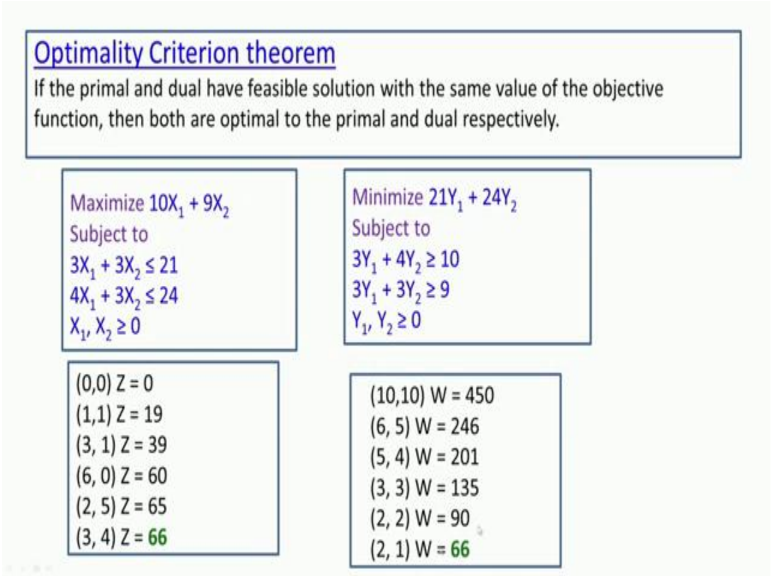

Then, we looked at the optimality criterion theorem and for the first time, we tried to relate the optimum solutions of both of them. And said that, if we are able to find a feasible solution to the primal and another feasible solution to the dual and if it so happens that the objective function values are equal, then they are optimum to the primal and dual respectively.

Now, we move on to a main duality theorem, which says, if the primal and dual have feasible solutions, independently both of them have physical solutions, then both will have optimum solutions with the same value of the objective function. Now, we need to understand this with respect to the optimality criterion theorem. The optimality criterion theorem also means something very similar, but the optimality criterion theorem looks at it from a different angle.

It says if I am able to find a feasible solution to the primal and to the dual and the objective function values are equal, then they are optimal. The main duality theorem says, if I am able to find a feasible solution to the primal and another feasible solution to the dual, they may not have the same value of the objective function. For example, if I am looking at this feasible solution to the primal with 19 and I am looking at this feasible solution to the dual with 135.

Let us say I have not evaluated any of these, I have not evaluated this, I have not evaluated this. So, right now my information is, I have a feasible solution to the primal and I have a feasible solution to the dual, the objective function values are different. But, based on these two alone, I can say that, now both primal and dual will have optimum solutions. Not only that, the value of the objective function will be the same; that is our main duality theorem.

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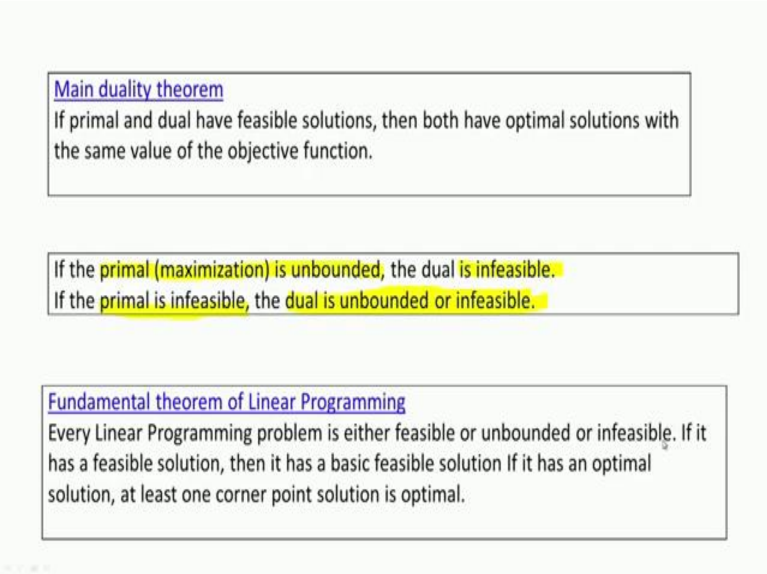

Now, this leads us to something, now if the primal is a maximization problem and we have already learnt that every linear programming problem. Only three things will be there, it will either give a solution or it will be unbounded or it will be infeasible. So, if we have a maximization problem, which is unbounded, which means, the objective function value for feasible solutions can go up to infinity.

How does a weak duality theorem behave? Now, when the weak duality theorem will have to say that, if there are feasible solutions to the dual, then every feasible solution to the dual should have an objective function value, which is greater than every feasible solution to the primal. And primal can have feasible solutions with values going up to infinity.

So, now what happens is, therefore we will say that if the primal maximization is unbounded, then obviously, we cannot have a feasible solution to the dual. Because, every feasible solution to the dual should now have objective function values greater close to infinity, which is not possible, then the dual becomes infeasible. So, when the primal is unbounded, primal maximization is unbounded, then the dual minimization is infeasible.

Now, the other thing that can happen, if the primal itself is infeasible, what will happen to the dual. If the primal maximization is infeasible, logically we would say that the dual will become unbounded. But, what actually happens is, the dual can become either unbounded or it can become infeasible. So, both these things can happen, when the primal which is maximization is increasing. We then move on to some other aspect of this problem, which is here.

Now, we look at the fundamental theorem of linear programming, which relates all the things that we have learnt till now. Every linear programming problem is either feasible or unbounded or infeasible, which is what I was mentioning only three things can happen, it is either feasible or unbounded or infeasible. If it has a feasible solution, then it has a basic feasible solution.

Now, out of the three cases, it will have a feasible solution in the feasible case, it will have a feasible solution in this case, it will also have a feasible solution in this case. If we realize carefully in both these cases, there was at least one basic feasible solution, there was at least one corner point. So, if it has a feasible solution, then it has a basic feasible solution, if it has an optimum solution, then at least one corner point solution is optimum or at the moment, we will say the optimum solution is a corner point solution.

The only other case, where multiple corner point solutions can be optimum is when the objective function line is parallel to anyone of the constraints. So, in a two variable case, so when it becomes parallel, then when it leaves the graph, then it will not go through a corner point, it will go through the line itself and therefore, we will have multiple optimum. So, that would give us a case with more than one corner point solution being optimum.

So, the general idea is, if it is feasible, then it is basic feasible, which means, it has a corner point solution and if it is infeasible, then obviously, there is no feasible region. So, this is called the fundamental theorem of linear programming. The most important understanding is, every linear programming problem is either feasible or unbounded or infeasible, only these three things will happen.

And then, we also have to understand these two results which says that, if the primal maximization is unbounded, then the dual is infeasible minimizations. Infeasible, if the primal maximization is infeasible, the dual is either unbounded or it is infeasible. So, this is another thing that we need to understand.

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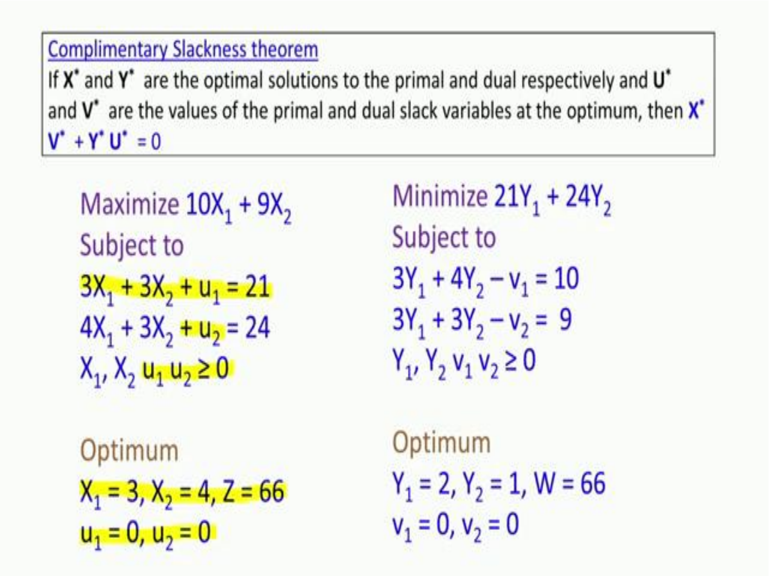

Now, we move on to another important aspect, another important theorem, which is called the Complimentary Slackness theorem. Now, what is this complimentary slackness theorem? We have introduced some notation to this problem. If X star and Y star are the optimum solutions to the primal and dual respectively and U star and V star are the values of the primal and dual slack variables at the optimum. Then, X star into V star plus Y star into U star is equal to 0 is generally state it as x v plus y u equal to 0.

Now, we need to introduce these notations, star is always used to represent the optimum solution. So, X star is the optimum solution to the primal, to begin with we will assume that the primal is the maximization, Y star is the optimum solution to the dual, u and v are the corresponding slack variables. Now, first we will take an example and explain the complimentary slackness theorem and then, we will also explain, how the complimentary slackness theorem can be used to find the optimum solution.

So, we start with our problem 10 X 1 plus 9 X 2, subject to 3 X 1 plus 3 X 2 less than or equal to 21. So, we would have written it earlier as 3 X 1 plus 3 X 2 plus X 3 equal to 21, 4 X 1 plus 3 X 2 plus X 4 equal to 24. Now, instead of writing X 3 and X 4, we write u 1 and u 2. To indicate that u 1 is the slack variable for the first constraint and u 2 is the slack variable for the second constraint. So, u 1, u 2 greater than or equal to 0 and we will have 0 u 1 plus 0 u 2.

So, just to be consistent with u 1 will be the slack variable associated with the first constraint, u 2 will be the slack variable associated with the second constraint and we already know that the optimum solution is X 1 equal to 3, X 2 equal to 4 with Z equal to 66. From simplex terms, X 1 and X 2 were basic, they were in the solution, X 3 and X 4 were non basic, they were not in the solution, they were fixed at 0. So, X 3 X 4 are fixed at 0, therefore, u 1, u 2 are 0.

Another way to look at it is, if X 1 equal to 3 and X 2 equal to 4, go back and substitute in the first equation. So, 3 into 3, 9 plus 3 into 4, 12 is 21, therefore, u 1 will become 0, 4 into 3, 12 plus 4 into 3, 12 is equal to 24, therefore, u 2 is equal to 0. So, the optimum solution to the primal will now have X star X 1, X 2 as 3 and 4 and U star, u 1, u 2 as 0 and 0. Now, let us write the dual of this problem.

Now, we know the dual of this problem, we have seen this dual so many times already. So, the dual is to minimize 21, Y 1 plus 24, Y 2 we would have written 3 Y 1 plus 4 Y 2 greater than or equal to 10, would have become 3 Y 1 plus 4 Y 2 minus X 3 equal to 10, 3 Y 1 plus 3 Y 2 minus X 4 equal to 9. Now, instead of writing X 3 and X 4 or Y 3 and Y 4, so 3 Y 1 plus 4 Y 2 minus Y 3 equal to 10, we would have written, 3 Y 1 plus 3 Y 2 minus Y 4 equal to 9, we would have written.

So, instead of writing Y 3 and Y 4 as the slack variables, we write v 1 and v 2 as the slack variables. So, v 1 is the slack variable associated with the first dual constraint and v 2 is the slack variable associated with the second dual constraint. Now, based on our discussion on the optimality criterion theorem, we now know that 2 comma 1 is the optimum solution, because 2 comma 1 is feasible, 2 into 3 is 6, 6 plus 4 is 10, 3 into 2, 6 plus 3 is 9 feasible. It has the same value of the objective function as that of another solution to the primal, therefore, it is optimum.

So, now, when Y 1 equal to 2 and Y 2 equal to 1, we go back and substitute. So, 3 into 2 6 plus 4, 10 will put v 1 is equal to 0, 3 into 3, 6 plus 3, 9 will put v 2 equal to 0. So, we have now independently written the optimum solution to the primal and the optimum solution to the dual, now we go back to the complimentary slackness. X star and Y star are the optimum solutions. So, X star is X 1 equal to 3, X 2 equal to 4, Y star is Y 1 equal to 2, Y 2 equal to 1, U star and V star are the values of the slacks.

So, u 1 equal to 0, u 2 equal to 0, v 1 equal to 0, v 2 equal to 0, then X into v plus Y into u is equal to 0. So, what is X into v, we have X 1 and X 2 in the primal, we have v 1 and v 2 in the dual, we have u 1 and u 2 in the primal, Y 1 and Y 2 in the dual. So, X into v means X 1, v 1 plus X 2, v 2, Y into u means, Y 1, u 1 plus Y 2, u 2. Now, X 1, v 1 plus X 2, v 2 plus Y 1, u 1 plus Y 2, u 2 is equal to 0 and X 1, X 2, Y 1, Y 2, u 1, u 2, v 1, v 2 are all greater than or equal to 0, therefore, individual terms should be equal to 0.

So, X 1, v 1 is equal to 0, X 2, v 2 is equal to 0, Y 1, u 1 is equal to 0, Y 2, u 2 is equal to 0; that is what the complimentary slackness tells us. Now, let us go and check, X 1 is equal to 3, v 1 is equal to 0, so X 1 v 1 is 0, X 2 equal to 4, v 2 equal to 0, X 2, v 2 is 0, Y 1 equal to 2, u 1 equal to 0, Y 1, u 1 is 0, Y 2 is equal to 1, u 2 equal to 0, Y 2, u 2 is 0. So, we see that at the optimum the complementary slackness conditions are satisfied.

In the next class, we will see some more aspects of the complimentary slackness theorem and some more results and interpretations of duality in the next module.