Introduction to Operations Research
Prof. G. Srinivasan
Department of Management Studies
Indian Institute of Technology, Madras

Module - 04
Dual
Lecture - 19
Duality Theorems

In this class, we look at some results and theorems which essentially relate the primal and the dual. So, we now move closer to answering the question, what is the importance of the dual, how is the dual related to the primal and what more can we learn about the dual that would help our understanding of linear programming better. So, we first start with a result which is called a Weak Duality Theorem.

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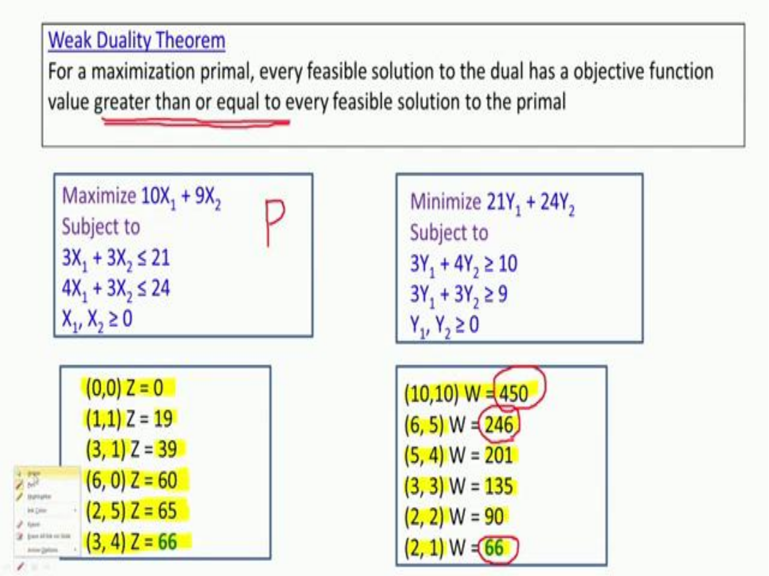

Now, let me read out the weak duality theorem and try to explain to you. For maximization primal, every feasible solution to the dual has an objective function value greater than or equal to that of every feasible solution to the primal. So, let me repeat again, for maximization primal, so we are assuming that the primal is a maximization problem. Every feasible solution to the dual of this problem will have an objective function value greater than or equal to that of every feasible solution to the primal.

So, let us try to illustrate or understand this theorem through some examples. So, let us start with the familiar example, the same problem that we have been solving to understand many things about linear programming. So, it is a maximization primal, so 10 X 1 plus 9 X 2 subject to 3 X 1 plus 3 X 2 less than or equal to 21, 4 X 1 plus 3 X 2 less than or equal to 24, X 1, X 2 greater than or equal to 0.

So, we do have a maximization primal, so this is our primal, so we write this as our primal, so we say this is our primal. Now, what does the theorem say? The theorem says that every feasible solution to the dual will have objective function values greater than or equal to every feasible solution to the primal. So, let us first find out some feasible solutions to the primal. Now, what is a feasible solution? Any X 1, X 2 that satisfies 3 X 1 plus 3 X 2 less than or equal to 21, 4 X 1 plus 3 X 2 less than or equal to 24, X 1, X 2 greater than equal to 0 is called a feasible solution.

Please remember that we also defined what are called basic feasible solutions and these basic feasible solutions were the corner points. But, we do have feasible solutions, any X 1, X 2 that satisfies will be a feasible solution. So, let us try to write down some feasible solutions and calculate the value of the objective function. So, the easiest one that we can think of is 0 comma 0, now 3 into 0 plus 3 into 0 is less than 21, similarly 0 is less than 24, X 1, X 2 greater than or equal to 0.

So, 0 0 is our first feasible solution that we can think of and 0 0 is feasible. So, when we substitute for 10 X 1 plus 9 X 2, we got objective function value as 0. Now, look at 1 comma 1 and check whether 1 comma 1 is feasible. Now, 1 comma 1 is feasible because 3 plus 3 is 6 which is less than 21, 4 plus 3 is 7 which is less than 24, 1 comma 1 is greater than or equal to 0 both of them are therefore, 1 comma 1 is feasible. 1 comma 1 is not a corner point it is not basic feasible, but it is feasible, because it satisfies all the constraints. So, 1 comma 1 is feasible, so when we substitute we get 10 plus 9, 19 for this.

Now, we try and increase and see whether we get better values of the objective function. So, let us try a solution which is 3 comma 1, so go back and substitute 3 comma 1, so 3 into 3 9, 9 plus 3 12 is less than 21, 4 into 3 12, 12 plus 3 15 is less than 24 both 3 and 1 are greater than or equal to 0 therefore, 3 comma 1 is feasible. So, we substitute 10 into 3 30 plus 9 gives us 39, so we have now not got 3 feasible solutions to the primal.

Now, we try 6 comma 0 make it slightly more, we try 6 comma 0 and observe that 6 into 3 18 is less than 21, 6 into 4 24 is equal to 24. So, 6 comma 0 is also feasible and gives a value 60. Now, we try another solution which is 2 comma 5, 2 comma 5 would give us 3 into 2 6 plus 5 into 3 15 equal to 21 and 4 into 2 8 plus 5 into 3 15 is equal to 23 which is less than 24, it is feasible. It is not basic feasible, it is feasible and it has an objective function value of 10 into 2 20 plus 9 into 5 45 equal to 65.

Finally, we look at 3 comma 4 and we realize that 3 into 3 plus 4 into 3 9 plus 12 is equal to 21, 4 into 3 12 plus 12 is equal to 24, this is also a feasible and it has an objective function value of 66. Now, for the sake of illustration, if we look at a solution 10 comma 0, now we realize that 10 comma 0 would have given us an objective function value of 70 which is bigger, but 10 comma 0 is not feasible, because 3 into 10, 30 is greater than 21.

So, we have now given a set of feasible solutions to the primal, there are infinite such feasible solutions to the primal. Now, we go on to write the dual of this problem, so we have already written the dual of this problem, there are two constraints. So, dual will have two variables, so the dual will try to minimize 21 Y 1 plus 24 Y 2 subject to 3 Y 1 plus 4 Y 2 greater than or equal to 10, 3 Y 1 plus 3 Y 2 greater than equal to 9, Y 1, Y 2 greater than or equal to 0, so this is the dual.

Now, let us find out some feasible solutions to the dual, first let us check whether 0 comma 0 is actually feasible. Now, we check 0 comma 0 we realize 3 into 0 plus 4 into 0 is 0, it is not greater than 10 therefore, 0 comma 0 is not feasible. So, we do not include it here, now we try some large values, let us say Y 1 equal to 10, Y 2 equal to 10. So, Y 1 equal to 10, Y 2 equal to 10 would mean 30 plus 40 70 which is greater than 10, 30 plus 30 60 which is greater than 9, 10 comma 10 is greater than or equal to 0 therefore, the value will be 210 plus 240 which is 450, so I would get 450 for this.

Now, I try to see whether I can get some other solutions better solutions, because it is a minimization problem. I will now try solutions with lesser objective function value. So, I try 6 comma 5, now go back and verify 6 into 3 18 plus 5 into 4 20 is 38 which is greater than 10, 6 into 3 18 plus 5 into 3 15, 33 is greater than 9. So, it is feasible the objective function is 21 into 6 plus 24 into 5 which would give us 246.

Now, we try some other solutions, say 5 comma 4 and by substituting into the two constraints we will realize that 5 comma 4 satisfies both the constraints. Therefore, it is feasible it would give us 201, similarly 3 comma 3 will would give us 9 plus 12 21, 9 plus 9 18. So, feasible 3 comma 3 would give us 135 45 into 135. Now, we look at 2 comma 2 we observe that 14 for the first one and 12 for the second one 2 comma 2 is also a feasible with value equal to 90. We try 2 comma 1, 2 comma 1 is also a feasible 6 plus 4 is 10, 6 plus 3 is 9 and we get 6 plus 6.

Now, we have given a sample set of feasible solutions to the dual. Now, with this set of feasible solutions to the primal and feasible solutions to the dual, let us see what happens to the weak duality theorem. We also have given only a sample, this is not certainly all the solutions, because there are infinite solutions to the feasible solutions to the primal and infinite feasible solutions to the dual. But, with this small set of solutions let us look at the weak duality theorem, every feasible solution to the dual has an objective function value greater than or equal to every feasible solution to the primal.

So, if we take the first one that is feasible to the dual, the objective function value is 450. Now, this 450 is bigger than all of these, similarly 246 is also greater than or equal to all of these. In the same way the smallest of these that we have got which is 66 is also a greater than or equal to all of these, it is also important to understand that it says greater than or equal to. So, we may have the solution that is equal to, but we cannot have solution that is greater than, the primal's value will not be greater the dual will be greater than or equal to it will not be less than that of any of the primal.

So, this is the weak duality theorem, it is a very important theorem for many reasons, for many things that are going to fall. Now, incidentally if we look at it very, very carefully, the way we constructed the dual we have actually used the weak duality theorem in mind. We started finding an upper estimate of the objective function for every feasible solution. Therefore, automatically the every feasible solution to the dual for a maximization primal will have an objective function value greater than that of the optimum solution of the primal.

Primal being a maximization problem, the optimum solutions objective function value will be greater than or equal to every feasible solutions objective function value. Therefore, we will realize that every feasible solution to the dual will have an objective function value greater than or equal to that of every feasible solution to the primal. So, this theorem is a very important theorem, it is called the weak duality theorem.

This theorem is usually represented this way for a maximization primal every feasible solution, it is also possible to say it in a different way for a minimization primal then every feasible solution to the primal will have an objective function value greater than or equal to everything of that corresponding maximization dual. The idea is the minimization problem whether it is a primal or a dual, every feasible solution to that minimization problem will have an objective function value greater than or equal to that of every feasible solution to the corresponding maximization problem, when the dual is written.

The primal maybe minimization dual maybe maximization it could be the other way, but essentially the minimization problem will have feasible solutions with greater than or equal to objective function value, this is the essence of the weak duality theorem, we now move on for some more results on duality.

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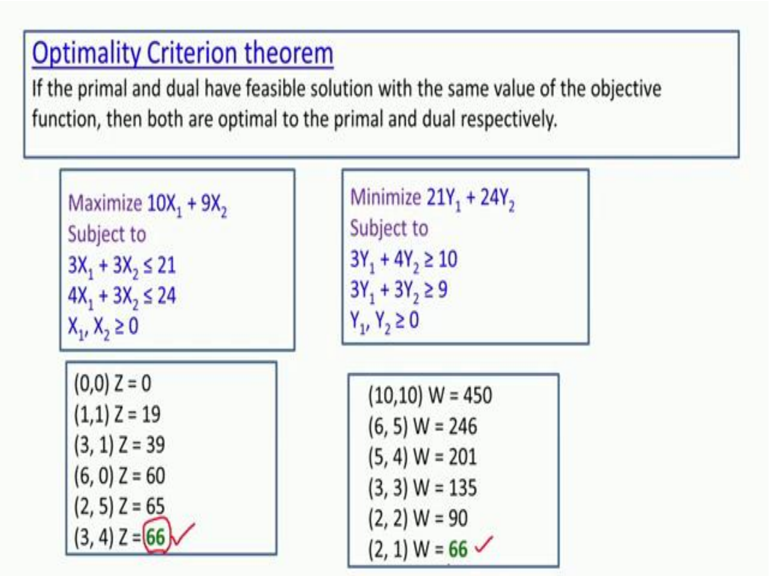

So, we look at another result called optimality criterion theorem, this is the second result which says if the primal and dual have feasible solutions with the same value of the objective function then both are optimal to the primal and dual respectively. So, now, for the first time we are trying to relate the optimum solutions of the primal and the dual. In the weak duality theorem we try to relate the feasible solutions of the primal and the dual, in this theorem we are trying to relate the optimum solution in a certain way.

Now, let us explain the optimality criterion theorem this way, so we go back to the same linear programming problem 10 X 1 plus 9 X 2 subject to 3 X 1 plus 3 X 2 less than equal to 21, 4 X 1 plus 3 X 2 less than equal to 24. Now, we go back to the dual which is written this way minimize 21 Y 1 plus 24 Y 2 and so on. Now, let us look at the same set of feasible solutions that we saw which we looked at when we studied the weak duality theorem, the same set of feasible solutions that we have.

Now, it started with 0 and went up to 66, now at the moment we will assume that this solution 66 is also a feasible solution to the given maximization problem and we will assume at the moment that we do not know the optimum solution to the maximization problem. So, let us assume that I have written the primal and I am writing down a set of feasible solutions, I have also written the dual and I am writing a set of feasible solutions to the dual, so we have shown the same set.

Now, I observe that here I have a solution to the primal which is feasible, which has an objective function value of 66 and I also have a different feasible solution to the dual which also has an objective function value of 66. Now, the optimality criterion theorem tells me that this solution 3 comma 4 is optimum to the primal and the solution 2 comma 1 is optimum to the dual. Why is this optimality criterion theorem true? It is true because if it is not so then the weak duality theorem will be violated.

If a solution with 66 is not optimum to this primal, then the optimum to this primal should have an objective function value more than 66. Now, if that is true, then the weak duality theorem is violated, because it says every feasible solution to the dual should have greater than or equal to. So, if this 66 is not optimum. Now, let us say for the sake of discussion 67 is optimum. Then, I have a feasible solution to dual with 66 and feasible solution to dual with 67 which would violate the weak duality theorem.

Therefore, by the weak duality theorem we can say that if I am able to find a feasible solution to the primal and I am able to find a feasible solution to the dual and they have the same value of the objective function, then there are optimal to the primal and dual respectively. So, the optimality criterion theorem is the first step in trying to relate the optimum solutions to the primal and the dual.

Now, let us assume that while searching for these solutions we are not getting equal solutions. I am still getting all solutions in this set which are greater than all solutions in this set, then what do we do how do I understand and relate the optimum solution to the dual that aspect we will see in the next class.