Introduction to Operations Research
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Module - 04
Dual
Lecture - 03
Writing Dual for a General LP (Continued)

In the last class, we were looking at the Dual of this particular minimization problem. When we began our discussion on the dual, we took a maximization problem with all less than or equal to constraints and all variables greater than or equal to 0. And the dual was a minimization problem with all greater than equal to constraints and all variables greater than or equal to 0. Now, we then post the next question of, if I have a problem with different types of constraints and different types of variables how does the dual look like.

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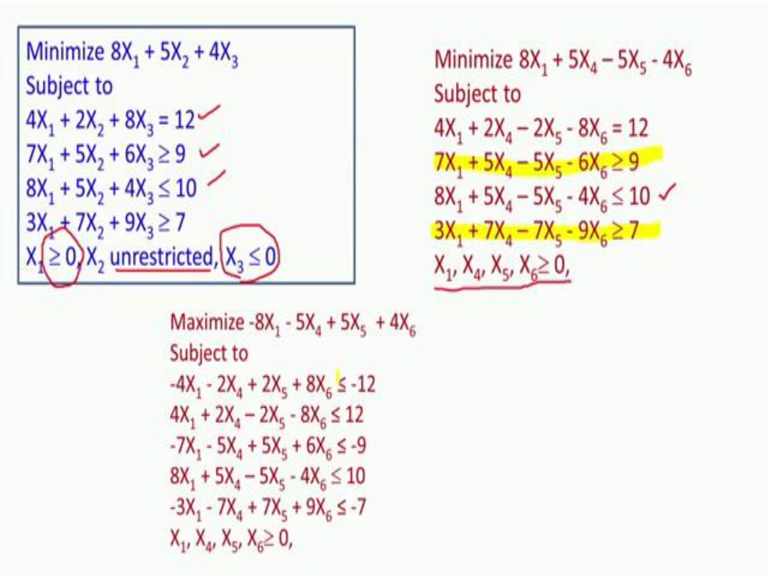

So, to explain that we looked at this problem which is a minimization problem with three different types of constraints that we have here, an equation less than or equal to and greater than or equal to and we also had three types of variables a greater than or equal to variable and unrestricted variable and a less than or equal to variable. Now, we said that we have to bring this to a form which is comfortable to us, which is a form of maximization with all constraints less than or equal to and all variables greater than or equal to.

So, we first redefine these variables such that they become greater than or equal to type. So, X 2 was redefined as X 4 minus X 5, X 3 was redefined as minus X 6 and we had a problem which was written like this. Now, we realize that this 5 X 2 has become 5 X 4 minus 5 X 5 and so on, this 8 X 3 has become minus 8 X 6, now X 6 is greater than or equal to 0, X 4, X 5 are also greater than or equal to 0. Now, we have brought it to the form where all the variables are of the required type which is greater than or equal to type.

Now, we have to bring it to the form where all the constraints are of the required type. So, what we did was we retained this less than or equal to constraint, we multiplied here with the minus 1 entire constraint with the minus 1, so that we get a less than or equal to. The 9 would become minus 9, we multiplied this also with minus 1, so that we get less than or equal to. The equation was written as two constraints, the same left hand said greater than or equal to 12, the same left hand side less than or equal to 12, we retained the less than or equal to, we multiply the greater than or equal to by minus 1.

So, when we did all that we got this form, now we wrote the primal in a form where all the variables are less than or equal to. Now, all the variables are less than or equal to, all the variables are greater than or equal to and all the constraints are less than or equal to. Now, we know to write the dual to a primal of this form and then, we wrote the dual to the primal of this form. But, meanwhile because we brought all the variables to greater than or equal to, we added one more variable the unrestricted resulted in an extra variable.

We brought all the constraints to less than equal to, so the equation resulted in an additional constraint, there was one greater than equal to one less than equal to. So, the origin and we of course, multiply the objective function with minus 1 to make it as maximization. So, the original problem has 3 variables and 4 constraints, now the modified problem has four variables and five constraints.

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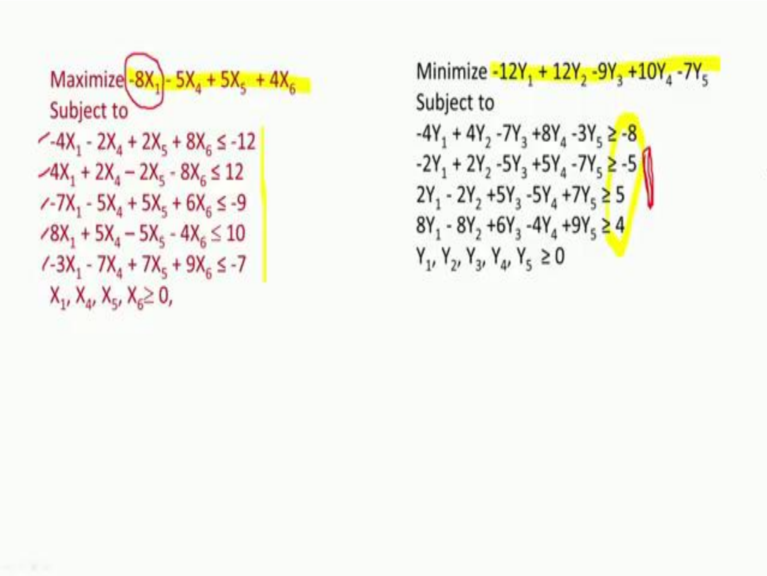

So, now we write the dual of this problem, I have shown the same problem here, now we can write the dual. We will now introduce dual variables Y 1, Y 2, Y 3, Y 4 and Y 5 there will be five dual variables, there will be four dual constraints corresponding to the four primal variables. These right hand side values will become objective function values, these objective function values will become right hand side values.

So, the dual will look this, the dual is now a minimization problem with five variables Y 1, Y 2, Y 3, Y 4 and Y 5. With the right hand side values of this, with these right hand side values becoming the values of the objective function. Now, these objective function values became the right hand side values, the transpose of the matrix repeated. For example, the first constraint would be minus 4 X 1 minus 4 Y 1 plus 4 Y 2 minus 7 Y 3 plus 8 Y 4 minus 3 Y 5 greater than or equal to minus 8, so we can write the dual this way.

Now, having written the dual this way, we realize that the dual now has five variables and four constraints. The original problem had only three variables and four constraints therefore, the dual should have four variables and three constraints plus the original problems objective function should be the right hand side of the dual and so on. So, we now have to modify this and bring it to a form which is consistent with the earlier table that we have seen.

So, first thing that we observe from this is, we look at these two constraints, you realize that you can multiply the first constraint with the minus 1, we will get 2 Y 1 minus 2 Y 2 plus 5 Y 3 minus 5 Y 4 plus 7 Y 5 is less than or equal to 5. And the same thing is now greater than or equal to 5 as another constraint therefore, we can merge these two constraints to get an equation.

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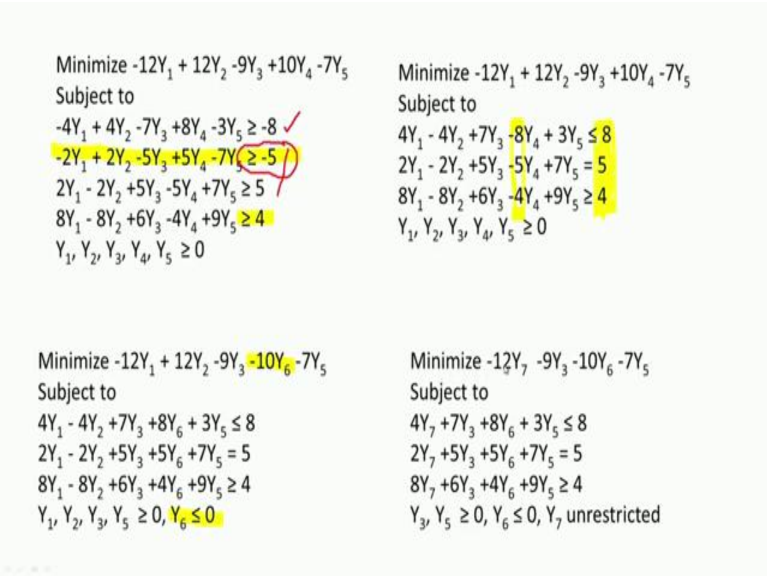

So, this is the dual that we had written, so this dual has five variables and four constraints. So, the first observation that we have is, we have this, the first constraint of the dual has a minus 8 in the right hand side. So, we try to bring it to a plus 8 by multiplying this constraint with the minus 1. So, that the right hand side values is positive, second observation that we have is these two constraints are similar. So, if we multiply this first constraint with the minus 1, we would get 2 Y 1 minus 2 Y 2 plus 5 Y 3 minus 5 Y 4 plus 7 Y 5 less than or equal to 5.

The next thing has the same left hand side greater than or equal to 5, you can merge these two and make them into an equation. So, we now do these two things, so we have now converted the first constraint this way by multiplying it with the minus 1. So, 4 Y 1 minus 4 Y 2 plus 7 Y 3 minus 8 Y 4 plus 3 Y 5 is less than or equal to 8 and as I mentioned with this constraint, this constraint is multiplied with minus 1 made into a less than equal to and then merged with the next one to give 2 Y 1 minus 2 Y 2 plus 5 Y 3 minus 5 Y 4 plus 7 Y 5 equal to 5. This constraint is retained as it is, because the right hand side value is plus 4, so we have plus 4, it is retained.

So, now what we have achieved is, we have done one thing where the right hand side values here are 8, 5 and 4 with the plus sign and these are exactly the objective function values of the given original problem. Then, when we take a look at this, we realize now that if we look at this variable Y 4, we realized that all three coefficients are negative for the variable Y 4 and Y 4 is greater than or equal to 0. So, we can now replace Y 4 by another variable call it Y 6 and then, we can make this variable less than equal to 0.

So, that the objective function the constraint coefficients are all positive. So, we do that next, now you observe that minus 8 Y 4 has become plus 8 Y 6 minus 5 Y 4 has become plus 5 Y 6 and so on, Y 6 has become less than or equal to 0, Y 6 has become less than or equal to 0 plus 10 Y 4 has become minus 10 Y 6. The other thing that we observe here is, we see 4 Y 1 minus 4 Y 2, 2 Y 1 minus 2 Y 2, 8 Y 1 minus 8 Y 2. Now, Y 1 minus Y 2 can be bunched together and can be made as an unrestricted variable.

So, we introduce a Y 7 which is an unrestricted variable and that becomes… Now, we have minimise minus 12 Y 7, because Y 1 minus Y 2 is the unrestricted variable minus 12 Y 7 minus 9 Y 3 minus 10 Y 6 minus 7 Y 5 and more importantly we have all the constraint coefficients now are the positive numbers. Now, the constraint matrix here of the dual is exactly the transpose of that of the given original problem, the right hand side values are the objective function value.

But, in the process Y 7 has being introduced as an unrestricted variable, now from here we introduce an unrestricted variable. So, the number of variables reduce by 1, we merge two constraints to an equation, where the number of constraints reduced by 1. Now, the dual has four variables and three constraints, the original primal had three variables and four constraints. So, the only thing that remains is, we have a minimization with all negative coefficients multiply with minus 1 to get the maximization and when we do that we get this.

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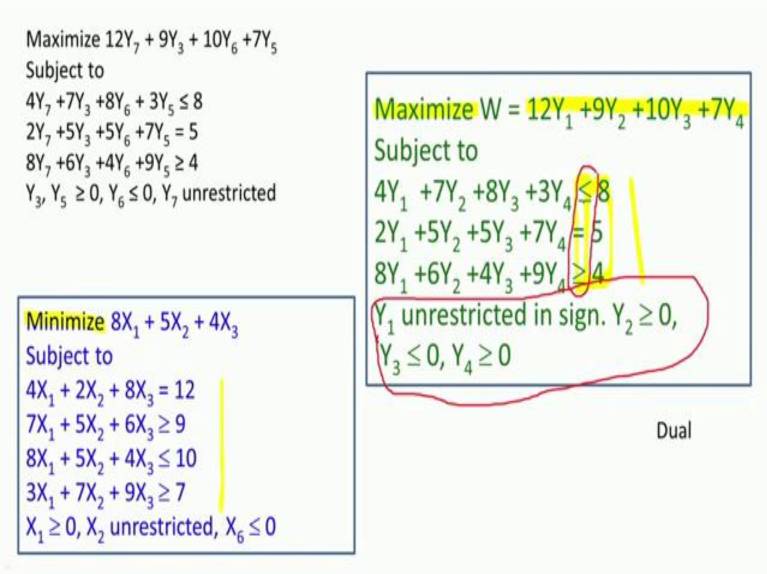

So, we have maximized 12 Y 7 plus 9 Y 3 plus 10 Y 6 plus 7 Y 5 and so on. Next thing we do is, we simply rename the variables to be consistent we call them as Y 1, Y 2, Y 3, Y 4 the variables are renamed. So, Y 7 becomes Y 1, Y 3 becomes Y 2, Y 6 becomes Y 3, Y 5 becomes Y 4 and it is rewritten this way. So, the dual is now a maximization problem with four variables and three constraints. Now, we go back and try to compare it with the original primal, the problem with which we started and the original primal is like this.

Now, we can do a mapping what we had seen earlier, we can now do that mapping we now realize that the original primal is a minimization problem it is dual is a maximization problem. The original primal has four constraints, the dual has four variables Y 1 to Y 4, original primal at three variables the dual has three constraints, the objective function values are 8, 5 and 4 the right hand side values are 8, 5 and 4. The right hand side values of the primal are 12, 9, 10 and 7 objective function is 12, 9, 10 and 7.

The given matrix is here the transpose is here 4, 2, 8 is the row 4, 2, 8 is a column, so we have this the same thing. The only difference is, we now have to see what happens to the sign of these how are these sign of these inequalities, how are these signs related to the primal and how are these signs related to the primal that is the only thing that remains that we have to see. So, what we can do is if we are given any primal we can simply write the primal is minimization, you can write a maximization, you can introduce the variables, you can do everything except that these three things have to be filled and these four things have to be filled. So, we need to find out what is the inequality or equation of the particular constraint and what is a sign of the particular variable that is, something which we need to do.

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Primal	Dual
max	Min
variables	Constraints
constraints	variables
RHS	Obj function
Obj function	RHS
A	A^T

~~Primal~~ (Maximization) D	~~Dual~~ (Minimization) P
1. $\leq$ constraint	$\geq$ variable
2. $\geq$ constraint	$\leq$ variable
3. equation constraint	unrestricted variable
4. $\geq$ variable	$\geq$ constraint
5. $\leq$ variable	$\leq$ constraint
6. unrestricted variable	equation constraint

Now, to do that let us revisit this again, this table we have already seen if the primal is a maximization, the dual is a maximization and vice versa. The number of variables of the primal is equal to number of constraints of the dual, number of constraints of the primal is equal to number of variables of the dual, right hand side of the primal becomes objective function of the dual, objective function of the primal becomes right hand side of the dual, the A matrix we constraints coefficient matrix becomes the transpose this we have already seen.

Now, this is a new thing that we are now going to see, now to see this we go back to the previous ((Refer Time: 15:34)), where the primal is a minimization problem, the dual is a maximization problem. Now, in the primal the first constraint is an equation, now remember there are four constraints here therefore, there are four variables here, there are three variables here, there are three constraints here. So, there is a relationship between constraint and variable and a relationship between variable and constraint.

So, we observe that the first constraint is an equation, the first variable becomes unrestricted in sign. The first the second variable is unrestricted in sign, the second constraint becomes an equation. So, there is a relationship between this equation and the unrestricted variable. So, if the i'th constraint is an equation in the primal, then the i'th variable will be unrestricted in the dual.

If the j'th variable is unrestricted in the primal, the j'th constraint will be an equation in the dual, this is also understandable. Because, for example, if we looked at this equation in the conversion process, we wrote this equation has two constraints. Now, these two constraints resulted in two variables in the dual which were subsequently merged into and unrestricted variable. So, an equation constraint results in an unrestricted variable, the unrestricted variable got expanded to two variables and then it became two constraints and these two constraints were merged into an equation.

So, the unrestricted variable corresponds to an equation constraint, so that relationship we can comfortably find out. The other thing is in the minimization, the first variable is greater than or equal to 0, in the maximization dual the first constraint is less than or equal to. In the minimization the third variable is less than or equal to, in the maximization the third constraint is greater than or equal to. In the minimization primal the second constraint is greater than or equal to, here the second variable is greater than or equal to, so that relationship we need to understand.

Now, that relationship we are trying to capture here, so if now whatever problem is given as a primal problem, in our example the minimization was the primal problem. So, this part is now becoming the primal for our discussion. So, this part is the primal for our discussion, this will be dual for our discussion, because our original problem was the given problem is a minimization problem. So, if the primal is a minimization problem if I have a greater than or equal to variable then the corresponding dual constraint will be less than or equal to constraint.

If I have a less than or equal to variable in a minimization primal the corresponding constraint will be a greater than or equal to. Similarly, if I have a greater than or equal to constraint in a minimization primal then the corresponding variable in the dual will be greater than or equal to. If I have less than or equal to constraint in the minimization primal, the corresponding variable will be less than or equal to. Alternately if the primal happens to be maximization and if less than equal to constraint then the corresponding variable in the dual will be a greater than or equal to variable.

Now, the way we explain is in the second constraint is a less than or equal to constraint in a maximization primal then the second variable will be a greater than or equal to variable in the minimization dual. So, if I have a greater than or equal to constraint in the maximization primal, the corresponding variable will be less than or equal to type. Similarly, if I have a greater than or equal to variable in a maximization primal the constraint will be greater than or equal in the minimization dual.

And if I have a less than or equal to variable in the maximization primal I will have a less than or equal to constraint in the minimization. Now, these four things we need to remember when we write the primal and the dual. So, one way is to remember them the other way is also to say that in any linear programming problem we have a greater than or equal to variable is a good variable with which we can start the simplex algorithm. For a minimization problem usually we find greater than or equal to constraints as the correct type of constraint and for a maximization problem we find less than equal to constraint as the correct type of constraints.

So, we can think the correct variable will match with the correct type of variable and the correct constraint will match with the correct type of constraint. For example, maximization problem less than or equal is a correct type of constraint. So, the greater than equal to variable is a correct type of variable that will come, for a maximization greater than or equal to is not a very correct type of constraint, usually you have less than equal to constraints there. So, minimization will have the opposite it will have a less than or equal to variable.

Similarly, greater than equal to variable is a correct type of variable, so greater than or equal to constraint for a minimization is a right kind of constraints. So, the correct kind of variable will map to a correct type of constraints and the not so desirable type of variable will map to a not so desirable type of constraints. So, this way using both these tables we will be able to write the dual for a given primal.

So, we now have to answer the next question having understood what is a dual and having learnt how to write the dual for a given primal, we have to ask the next question why is the dual important after all these two problems one comes from the other and how are they otherwise related. So, we now look at what is called primal dual relationships or understanding the dual further through some results and theorem which we will see in the next class.