**Introduction to Operations Research**
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**Module - 04**
**Dual**
**Lecture - 02**
**Writing the Dual for a general LP**

In this class, we will study the Primal and the Dual in little more detail. We will first try and write the dual for a given primal without going through the process of finding the upper estimate. And then we will define a way by which it can be written for any given linear programming problem.

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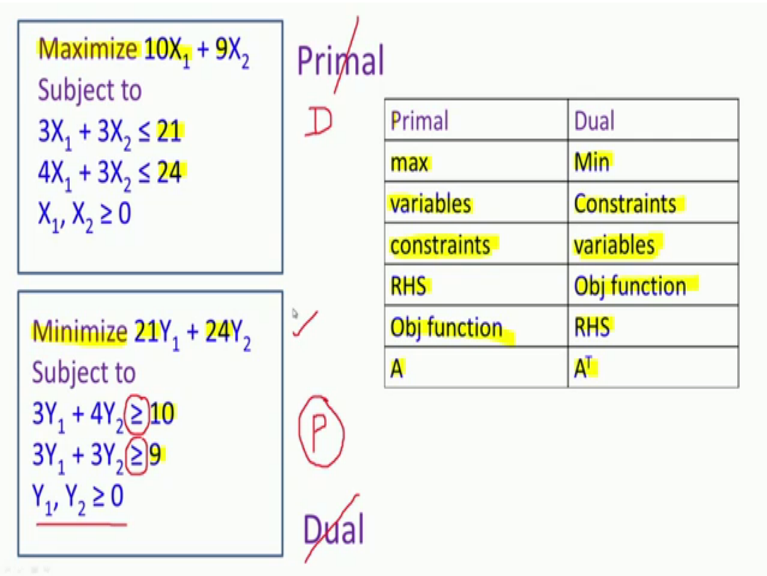


So, let us start with the same primal, the given problem is called primal. Maximize 10 X 1 plus 9 X 2 subject to 3 X 1 plus 3 X 2 less than equal to 21, 4 X 1 plus 3 X 2 less than equal to 24, X 1 X 2 greater than or equal to 0. In the last class, we wrote the dual of this problem and the dual of this problem is minimize 21 Y 1 plus 24 Y 2 subject to 3 Y 1 plus 4 Y 2 greater than or equal to 10, 3 Y 1 plus 3 Y 2 greater than or equal to 9, Y 1 Y 2 greater than or equal to 0.

Now, usually the variables X are used in the primal and the variables Y are used in the dual. It is not a reject condition or anything, just to understand that there are two different problems. Consistently, we use X for the primal and Y for the dual. One should also understand that it is not necessary that the primal is always a maximization problem, the given problem is called the primal. If the given problem is a minimization problem, then the primal is a minimization problem.

We will also see what happens to the dual of a minimization problem. Now, we make this table and we try to explain this table in the context of the problem that we have already looked at. So, if we look at this example, the primal is a maximization problem, primal is a maximization problem, so primal is a maximization problem. Now, we observed that the dual is a minimization problem, so the dual is minimization.

So, when the primal is a maximization problem, the dual is a minimization problem. Now, the primal has two variables X 1 and X 2, the dual has two constraints here. It is so happen that this primal has two variables and two constraints, so dual will have two variables and two constraints, but there is a relationship between a primal variable and a dual constraint. Because, when we wrote this we said we multiply the first one by A and the second one by B, such that 3 A plus 4 B is greater than or equal to 10.

And those A and B, finally became Y 1 and Y 2 and because we had two constraints here, we introduced two variables Y 1 and Y 2 for the dual. Therefore, the number of constraints in the primal will decide the number of variables in the dual and having define this Y 1 and Y 2, we went ahead and said 3 Y 1 plus 4 Y 2 is greater than or equal to 10, 3 Y 1 plus 3 Y 2 is greater than or equal to 9. Therefore, the number of constraints in the dual will be equal to number of variables in the primal, very important to understand that.

In this example, if we may not be able to bring it out very well, because both are 2 in number, but the next example that we will see, we will have an unequal number of variables and constraints and we will be able to bring this aspect much better. So, we need to understand that the primal has two variables X 1 and X 2, the dual has two constraints. The primal has two constraints and the dual will have two variables.

Because, for every constraint we introduce a variable, because of these two constraints we said let us have A and B, which later became Y 1 and Y 2. Now, we also observe that the right hand side values of the primal have become the objective function values of the dual. So, 21 and 24, 21 and 24, so right hand side value of the primal becomes the objective function value of the dual. The objective function values of the primal 10 and 9 will become the right hand side values of the dual.

So, objective function value of the primal becomes the right hand side value of the dual. Now, the coefficient matrix 3 X 1, 3 X 2, 4 X 1, 3 X 2, if this is represented in the form of a matrix, this will become 3, 3, 4, 3 and we realize that for the dual, it is become the transpose. So, 3, 4 was the first column, it will become the first row 3, 4. 3 3 is the second column that will become the second row, so the rows becomes the columns and columns becomes the rows. Therefore, this matrix is called A, then here in the dual it will become A transpose.

You would have learnt in algebra or matrices, as to what is meant by the transpose of a matrix. So, given a matrix the transpose is a another matrix such that it has, the columns is equal to the number of rows, the rows equal to columns and the rows automatically become the columns. So, 3, 3 which is the first row will become 3, 3 which is the first column and 4, 3 which is the second row will become 4, 3 which is the second column. So, this is the first step in primal dual relationship. So, now, without seeing this I can write the dual. I can say that, if the primal is maximized 10 X 1 plus 9 X 2 subject to this.

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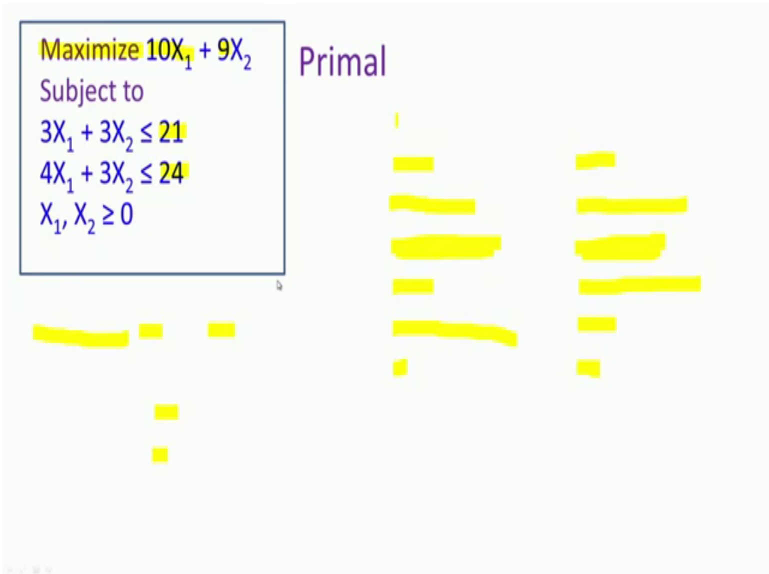


I will now go back and say that, now I will go back and say that if I have this primal, now if I want to write the dual, the primal has two constraints. So, dual will have two variables Y 1 and Y 2. So, when I have Y 1 and Y 2, I will say 3 Y 1 plus 4 Y 2 greater than or equal to 10, 3 Y 1 plus 3 Y 2 greater than or equal to 9. The objective function will be to… Because, this is maximization, objective function will be minimization, so objective will be to minimize 21 Y 1 plus 24 Y 2 and Y 1, Y 2 greater than or equal to 0 is what you get here.

So, from this relationship we understand if the primal has all this ((Refer Time: 07:37)), then the dual will have all this. I am also going to make a generalization that if for some reason, this is the given problem. If for some reason, this problem is the given problem and this problem becomes the primal. If this problem becomes the primal which I call as p, then let me try and write the dual in the same way. Now, this primal has two constraints therefore, it is dual will have two variables.

Now, let those variables we called X 1 and X 2. Now, 3 X 1 plus 3 X 2 is less than or equal to 21, 4 X 1 plus 3 X 2 is less than or equal to 24, maximize 10 X 1 plus 9 X 2, X 1 X 2 greater than or equal to 0. So, you observe that if this problem becomes the primal, then this problem will become the dual. So, we go back and generalize that if this is the primal, the primal is a minimization, dual is a maximization. Primal, number of constraints in the primal equal to number of variables in the dual and the same table is applicable.

Now, if we go back and if we realize what we have done, up to all this is fine or up to all this is you can write easily. The only place when we actually write a dual from the primal, the only place where we have to be careful ((Refer Time: 09:19)) and do it well is the sign of these inequalities that we have as well as the definition of the variables. The numbers are intact, this table tells us that the numbers are intact. If the primal has 10 and 9 here, the dual will have 10 and 9 in the right hand side.

If the primal has 21 and 24 here, dual will have 21 and 24 in the objective, if the primal has 3, 3, 4, 3 dual will have 3, 4, 3, 3. But, what happens to the sign, s I g n sign of these inequalities and if there are equations, what do I do. So, we should be able to write the dual of a general linear programming problem, so we explained that through another illustration.

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Minimize $8X_1 + 5X_2 + 4X_3$

Subject to

$4X_1 + 2X_2 + 8X_3 = 12$

$7X_1 + 5X_2 + 6X_3 \geq 9$

$8X_1 + 5X_2 + 4X_3 \leq 10$

$3X_1 + 7X_2 + 9X_3 \geq 7$

$X_1 \geq 0$, $X_2$ unrestricted, $X_3 \leq 0$

P

Minimize $8X_1 + 5X_4 - 5X_5 - 4X_6$

Subject to

$4X_1 + 2X_4 - 2X_5 - 8X_6 = 12$

$7X_1 + 5X_4 - 5X_5 - 6X_6 \geq 9$

$8X_1 + 5X_4 - 5X_5 - 4X_6 \leq 10$

$3X_1 + 7X_4 - 7X_5 - 9X_6 \geq 7$

$X_1, X_4, X_5, X_6 \geq 0,$

Maximize $-8X_1 - 5X_4 + 5X_5 + 4X_6$

Subject to

$-4X_1 - 2X_4 + 2X_5 + 8X_6 \leq -12$

$4X_1 + 2X_4 - 2X_5 - 8X_6 \leq 12$

$-7X_1 - 5X_4 + 5X_5 + 6X_6 \leq -9$

$8X_1 + 5X_4 - 5X_5 - 4X_6 \leq 10$

$-3X_1 - 7X_4 + 7X_5 + 9X_6 \leq -7$

$X_1, X_4, X_5, X_6 \geq 0,$

Here we consider a minimization problem as the primal, so this problem is a minimization problem which is the primal and this problem has three variables and four constraints. Now, we have an unequal number of variables and constraints. We have already seen that a linear programming problem can be either a maximization problem or a minimization problem, so this is a minimization problem.

We have already seen in our earlier formulations that the constraints can be equations, it can be greater than or equal to, it can be less than or equal to. But, so far in all our formulations we have seen only variables that are greater than or equal to, we have not seen the other two types of variables. So, in some situations, at least while formulating we give provision for some variables to be less than or equal to. So, in this formulation we have now assumed X 3 to be a less than or equal to variable.

Now, we have also said that X 2 is unrestricted, so a variable is unrestricted if it could take a positive value or a 0 value or a negative value. For the sake of illustration, if X 2 represents a profit or a net profit from buying and selling something, then the net profit could be positive or 0 or negative. So, a nice way of representing an unrestricted variable is the difference between two variables. Just as I explained, a net profit is a difference between selling and a buying, so an unrestricted variable is replaced by the difference of two variables.

So, with this in mind let us now write this primal further. We also observe that, in the earlier instance we were able to write the dual for a primal which has a maximization objective, all constraints less than or equal to type and all variables greater than or equal to type. Now, here we have a minimization and we know that we can convert this to maximization by multiplying with the minus 1 which we will do, but then we have mixed set of constraints and mixed set of variables.

So, we now have to bring all the variables to greater than or equal to and all the constraints to less than or equal to, so we first consider the variables. Now, X 1 greater than or equal to 0 is fine, the unrestricted variable is now replaced or substituted with difference of two variables which we call X 4 and X 5. So, X 2 becomes X 4 minus X 5 and X 3 which is less than equal to variable is now replaced by minus X 6, so that X 6 is greater than or equal to 0.

Now, we see what is the effect of this change? So, the effect of change is here. Now, all the variables have now become greater than or equal to 0, because X 2 has become X 4 minus X 5 and X 3 has become minus X 6, so that X 6 is greater than or equal to 0. And because X 3 is replaced with the minus X 6, now you see that the plus 4 X 3 has become minus 4 X 6 plus 8 X 3 has become minus 8 X 6 and so on. So, all these coefficients have become negative and this also has become negative.

X 2 is an unrestricted variable replaced by X 4 minus X 5, so 5 X 2 will become 5 X 4 minus 5 X 5, 2 X 2 will become 2 X 4 minus 2 X 5, 5 X 2 becomes 5 X 4 minus 5 X 5 and so on. So, the change is captured here, the problem continues to be a minimization problem, but since the unrestricted variable is substituted as a difference of two variables. We now have four variables which are X 1, X 4, X 5 and X 6, but all four of them are greater than or equal to 0, and therefore we have now brought it to a form, where all variables are greater than or equal to 0.

The next thing we have to do is to bring it to a form, where all the constraints are less than or equal to. Now, this constraint is already less than or equal to therefore, we are with this constraint. Now, this constraint is a greater than or equal to constraint and an easy way to make it less than or equal to is to multiply the left hand side and a right hand side by minus 1. Similarly, we can do this multiply with minus 1 the left hand side and right hand side, this will become a less than or equal to constraint, now the equation posses a tricky situation.

So, now the equation will be written as two constraints for example, 4 X 1 plus 2 X 4 minus 2 X 5 minus 8 X 6 equal to 12 can be written as 4 X 1 plus 2 X 4 minus 2 X 5 minus 8 X 6 greater than or equal to 12 and 4 X 1 plus 2 X 4 minus 2 X 5 minus 8 X 6 less than or equal 12. So, this will be written as two constraints, one with greater than or equal to 12, the other with less than or equal to 12. Now, the less than equal to 12 constraints, we will keep it as it is.

The greater than equal to 12 constraints, we will again multiply with the minus 1, so that it becomes less or equal to. Now, let us see the effect of all these changes. So, we now have this, now what have we done. The third constraint was 8 X 1 plus 5 X 4 minus 5 X 5 minus 4 X 6 equal to 10 is retained here. So, it is kept here, there is no problem with this constraint. Now, this is a greater than or equal to constraint therefore, multiply with minus 1, so if you get a less than or equal to, the plus 9 becomes minus 9 and you will see that all these coefficients have become negative, correspondingly negative and positive respectively.

Now, this is a greater than or equal to, we did the same thing minus 3 X 1 minus 7 X 4. So, multiply with the minus 1 becomes a less than or equal to, right hand side becomes minus 7, this minus 9 becomes plus 9, because we are multiplying with minus 1 and so on. The equation is written as two constraints 4 X 1 plus 2 X 2 minus 2 X 5 minus 8 X 6 less than or equal to 12, which is here and the same thing greater than or equal to 12.

Now, that is multiplied with a minus 1 to get a less than equal to minus 12 and the coefficients are changed. So, now, by doing this thing what we have ensured is that and what we have done is we have changed the minimization to a maximization by multiplying this with minus 1, so this is the maximization. So, now, we have brought the original problem into a maximization problem, now this original primal is now made into a maximization problem, where all variables are greater than or equal to type and all constraints are less than or equal to type, all constraints are less than or equal to type.

Now, in the process we introduced one more variable, because the unrestricted variable gave rise to another variable and we introduced one more constraint, because the equation here was written as two different constraints. So, the original problem had four constraints, now the modified problem has five constraints. The original problem had three variables, the modified problem has four variables, but we have now written it in a form, where we can write the dual. So, this is still the primal, but this is a modified primal, we can now write the dual of this.

Now, this primal has five constrains, so the dual will have five variables Y 1 to Y 5. This primal has four variables, the dual will have four constraints. Now, the dual will have all variables greater than or equal to 0 and all constraints of the greater than or equal to type, so we can write the dual of this.

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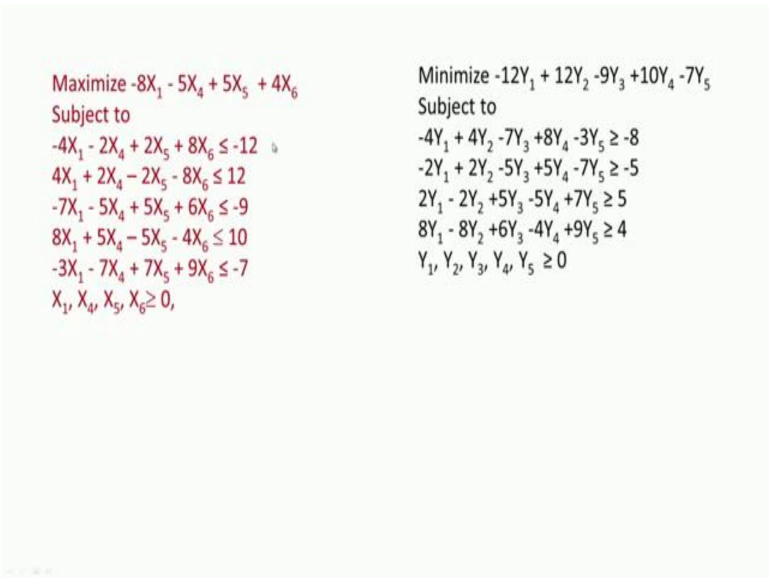


So, this is the new primal after we have make sure, I have written the same problem here, now we can write the dual of this. Now, the dual variables will be Y 1, Y 2, Y 3, Y 4, Y 5 for the five constraints, we will have them. Now, since the new primal is maximization, the new dual will be minimization. Now, the objective function value of the dual will be the right hand side of the primal, so minus 12 X 1 plus 12 X 2 minus 9 X 3 plus 10 X 4 minus 7 Y 5.

Y 1 to Y 5 are the dual variables, so dual will be to minimize minus 12 Y 1 plus 12 Y 2 minus 9 Y 3 plus 10 Y 4 minus 7 Y 5, which you can read from here. Now, we take it one variable at a time, minus 4 Y 1 plus 4 Y 2 minus 7 Y 3 plus 8 Y 4 minus 3 Y 5 will be greater than or equal to minus 8 which is written here. Similarly, all of them are written here, so this is the dual now. Now, this dual has four constraints and five variables, but if you go back to the original problem, the original problem had three variables and four constraints.

Therefore, this dual should have three constrains and four variables. So, we now reduce this dual into a form, where it has three constraints and four variables, we need to do that.

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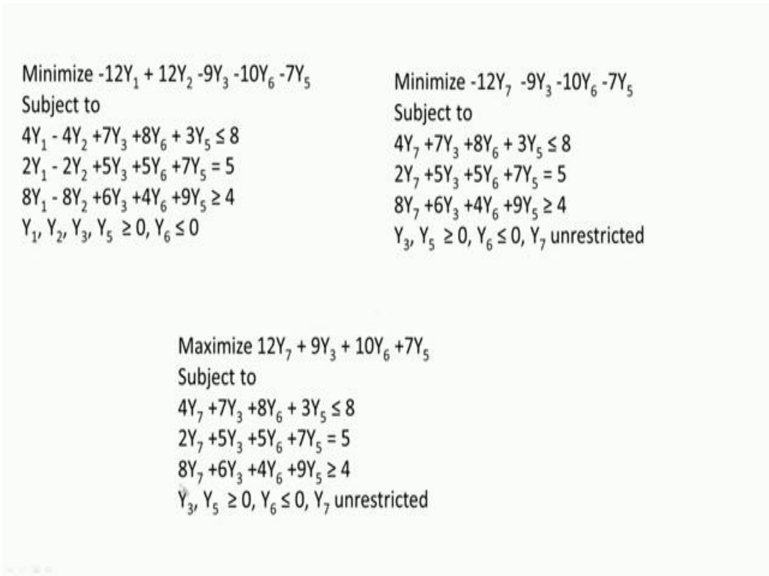


Now, let us do that here, I have written the same problem. So, this is what we have as the dual, now we need to do a few things. Now, if we see these two carefully, these two constraints carefully. If you multiply this constraint with a minus 1, we will get 2 Y 1 minus 2 Y 2 plus 5 Y 3 minus 5 Y 4 plus 7 Y 5 will be less than or equal to 5. So, this less than equal to 5 and greater than or equal to 5 will give us an equation, so we do that.

Now, that has happened here, so we do this that has happened here. Now, we also realize that, we observe here that, if we look at… So, this has been written, so here Y 6 is less than or equal to 0, so we look at the variable, we look at this variable Y 4. The first thing we do is, this is has a minus value on the right hand side, so multiply this with the minus 1, now this has become an equation. So, this will have 5, this will have 4, multiply this with the minus 1.

So, when we multiply this with the minus 1 we get this, now all these values have changed. Multiply this with the minus 1, 8, 5 and 4 have happened, now go back to the previous one. Now, when we multiply this with the minus 1 we would have got minus 8 here and this plus 5 would also have become minus 5 and this also has minus 4. So, for the variable Y 4 all the coefficients are negative. So, now, replace Y 4 with the minus Y 6, so that Y 6 becomes less than or equal to 0, now that is shown here.

Now, all the three constraints have 8, 5 and 4 and now, we realize that Y 4 has been replaced with minus Y 6. Therefore, Y 6 is less than or equal to 0 and the minus values have become plus values. Now, what happens is we have three constraints here and we have five variables and we now realize that 4 Y 1 minus 4 Y 2, 2 Y 1 minus 2 Y 2, 8 Y 1 minus 8 Y 2. So, Y 1 minus Y 2 now can be replaced by another variable which is unrestricted, so we do that next, we call it as Y 7. So, 4 Y 7 plus 7 Y 3 plus 8 Y 6 plus 3 Y 5, 4 Y 1 minus 4 Y 2 becomes 4 Y 7, Y 7 becomes unrestricted. Now, this minimization is now changed into a maximization which is the last thing that we need to do.

So, we now maximize 12 Y 7 plus 9 Y 3 plus 10 Y 6 plus 7 Y 5 subject to all of these. So, we have now written the dual for the given problem, the given problem was a minimization, the dual is a maximization. Now, we observe that the original problem had three variables, the dual has three constraints, the original problem had four constraints, the dual has four variables. We will see some more aspects of this primal dual relationship in the next class.