Introduction to Operations Research
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Module - 04
Dual
Lecture - 16
Motivation to the Dual

In this class, we introduce the Dual of a linear programming problem. We consider the maximization problem that we have seen before and then, we try to understand what is the dual? And how a dual is formed from a given linear programming problem?

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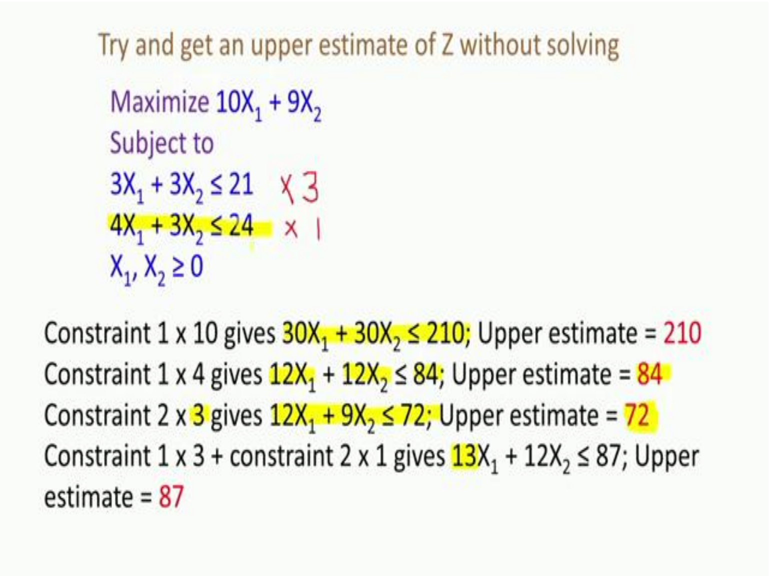

So, let us look at the same linear programming problem that we have been solving and try to get an upper estimate of the objective function without solving it. So, let me explain this with the example. We look at this problem which is to maximize 10 X 1 plus 9 X 2 subject to 3 X 1 plus 3 X 2 less than or equal to 21, 4 X 1 plus 3 X 2 less than or equal to 24, X 1 and X 2 greater than or equal to 0.

Now, at the moment let us assume that we have not solved this problem or we right now do not have a way to solve this problem. But, we are interested in finding out, what would be the upper estimate of the objective function without actually solving it. Now, we try and do the following things, so let me explain each of these one after another.

Now, let us take the first constraint 3 X 1 plus 3 X 2 is less than or equal to 21, now let us multiply the first constraint by 10.

So, we multiply the first constraint by 10. So, if we multiply the first constraint by 10, we will get 30 X 1 plus 30 X 2 will be less than or equal to 210. Now, this tells us something. What does this tell us? This tells us we also should have X 1, X 2 greater than or equal to 0. So, when X 1, X 2 is greater than or equal to 0, 30 X 1 plus 30 X 2 will be greater than or equal to 10 X 1 plus 9 X 2, because X 1 and X 2 are greater than or equal to 0.

Now, the first constraint multiplied by 10 says that 30 X 1 plus 30 X 2 has to be less than or equal to 210. Therefore, 10 X 1 plus 9 X 2 which itself is less than or equal to 30 X 1 plus 30 X 2 will have to be less than or equal to 210. Therefore, we can say without solving that the value of objective function at the optimum cannot exceed 210, this looks like a simple result. For example, if somebody says the optimum solution to this problem has an objective function value of 300, then based on this result we could say no we cannot have the objective function value of the optimum more than 210.

So, we understand up to that, so we have now got an upper estimate of the objective function without actually solving it by using these constraints. So, at the moment this 210 is an upper estimate of the objective function value, now let us do this. Now, instead of multiplying the first constraint by 10, let us multiply the first constraint 4 instead of multiplying the first constraint by 10, we multiply the first constraint by 4.

So, the first constraint is multiplied by 4, so if we multiply the first constraint by 4, we will get 12 X 1. Because, 3 X 1 plus 3 X 2 less than equal to 21, when we multiply by 4 will give us 12 X 1 plus 12 X 2 less than or equal to 84. Now, we again learn something from this. Now, what do we learn? As long as X 1, X 2 is greater than or equal to 0, 10 X 1 plus 9 X 2 will be less than or equal to 12 X 1 plus 12 X 2. Because, this coefficient 12 is greater than or equal to 10 and this coefficient 12 is greater than or equal to 9.

Therefore, 12 X 1 plus 12 X 2 will be greater than or equal to 10 X 1 plus 9 X 2. Now, if 12 X 1 plus 12 X 2 itself should be less than or equal to 84, then 10 X 1 plus 9 X 2 should be surely less than or equal to 84 and therefore, we can now say confidently, that the value of the objective function at the optimum will not be greater than 84 and 84 is now an upper estimate of the value of the objective function at the optimum.

Now, so far what have we done? Based on multiplying the first constraint by 10, we said the upper estimate is 210. Now, based on multiplying the first constraint by 4 we have said that the upper estimate is 84. Now, between the upper estimates of 210 and 84, 84 is more meaningful, because we can now say that the value of the objective function cannot exceed 84 and that is a more meaningful expression rather than saying, it cannot exceed 210.

So, what we have to do is, when we try and get an upper estimate the smaller the upper estimate is the more meaningful the upper estimate is. And then, we have also now understood that, if we multiply this equation or the constraint by a constant such that, we are able to get the after multiplication if we are able to get something such that, the coefficients are related, then we can get the upper estimate. Let us continue with this by doing something with the second constraint.

Now, let us multiply the second constraint by 3, so we multiply the second constraint by 3 to get 12 X 1 plus 9 X 2 is less than or equal to 72. Now, again going back, so long as X 1 and X 2 are greater than or equal to 0, 10 X 1 plus 9 X 2 will be less than or equal to 12 X 1 plus 9 X 2 and since, 12 X 1 plus 9 X 2 itself should be less than or equal to 72. 10 X 1 plus 9 X 2 has to be less than or equal to 72 and 72 is now an upper estimate to the value of the objective function.

Now, we have tried to calculate the upper estimate in three ways and we got 210, we got 84, we got 72. Now, 72 is more meaningful than 84 and 210, because we are able to get a lower value of the upper estimate. So, 72 is meaningful and we can now say that the value of the objective function at the optimum cannot exceed 72 without solving this problem. So, if somebody comes and says, the value is 75, you can say that it cannot be so, because our upper estimate is 72.

Now, how did we get this upper estimate? We got these upper estimates by simply taking a constraint and by multiplying the constraint with the positive, in this case a positive number. Because, if we multiply this with a negative number, the inequality sign will change, so, so far we have multiplied it with a positive number or we may generalize it and say by a non-negative number, by a positive number.

And by doing so, after multiplication, if the coefficients after multiplication are individually greater than or equal to that of the objective function, then we could say the right hand side obtained after the multiplication is an upper estimate. In this case we multiplied by 10, the right hand side happened to be 210, the individual coefficients were greater than or equal to 10 and 9. Similarly, when we multiplied by 4, the right hand side came to 84, the individual coefficients 12 and 12 were greater than or equal to 10 and 9 and we got this.

Similarly, for the third, using the second constraint the third time when we did this, we multiplied by 3 to get 12 and 9 which were greater than or equal to 10 and 9 and therefore, the upper estimate happen to be 72. So, we can now generalize and say that, we can take any one of the constraints and multiply this constraint by a positive number.

So, that the sign of the inequality does not change and by doing so, after multiplication if the coefficients are greater than or equal to those of the objective function taken for every variable one at a time, then after multiplication of the constraint whatever is the right hand side value is the upper estimate. Now, let us do one more thing. Now, instead of taking one constraint at a time, can we do the same thing with multiple constraints?

Now, in this example there are only two constraints therefore, we take both of them, if the problem had a larger number we could take a subset. So, if we take multiple constraints we take two of them. Multiply the first constraint by 3 and multiply the second constraint by 1 or simply add the second constraint. So, when we multiply the first constraint by 3, we get 3 into 3. So, what we do is, this is multiplied by 3 and this is multiplied by 1, which is the same as adding it.

We get 3 into 3 9, 9 plus 4 13 as the coefficient for X 1, now 3 into 3 9 plus 3 12 as the coefficient for X 2. Now, this 13 and 12 are greater than or equal to this 10 and 9 respectively. Therefore, the value 87 which was got by multiplying the first constraint by 3, 21 into 3 63 plus 24 into 1 gives us 87, now 87 is an upper estimate of the objective function value. Now, it so happen is that, by now we have a slightly better estimate of 72 therefore, one could go back and say, at the moment this 87 has not added too much value, because 72 is a better estimate.

But, the idea is important, rather than taking one constraint at a time, can I take multiple constraints at a time. Multiply each of these constraints by a positive or a non-negative number, so that the sign of the inequality does not change and then, add and after doing this multiplication and addition, if I will get a single constraint. For example, when we multiplied this by 3 and the second constraint by 1, we got 13 X 1 plus 12 X 2 less than equal to 87 and after all this, if the individual coefficients are greater than that of the objective function, then the right hand side of the new constraint will be the upper estimate of the objective function value.

So, when we multiply and we have to now suitably find out these numbers 3 and 1, 4 and 2, whatever it be such that, these individual coefficients will be greater than or equal to the corresponding individual coefficients. So, we have learnt one thing now instead of two. We started by taking one constraint at a time, but then now we have said that, if we take multiple constraints and we can do the same thing.

We can also show that, taking one constraint at a time is like taking more than one and giving a weight of 0 to some other constraint. So, we can take any subset of constraints, multiply them by a non-negative, this case positive number and add them. So, that the inequality the sign is not affected and after doing so, we will get a single constraint and if that constraint is such that, the left hand side coefficients are greater than or equal to that of the objective function coefficients variable by variable, then the right hand side is an upper estimate of the objective function value.

So, now, using these four we have come to a thing that 72 looks the best upper estimate. But, more importantly the idea behind the fourth one, where I can multiply the constraints by a certain non-negative number and add them is something that is important and we carry that further. So, what do we do next?

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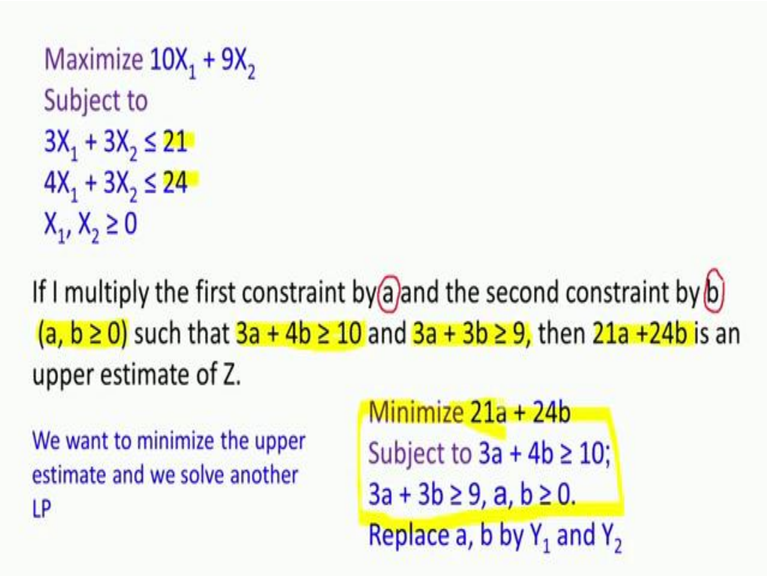

Now, we go back and say that, I have written the same problem again for reference. Now, I go back and say, instead of multiplying these constraints by some known numbers, we did 3 and 1. Now, I am going to say that if I multiply the first constraint by a, I multiply the second constraint by b, at the moment I do not know b and a and b. But, if I multiply them by a and b and I have to make sure that, a and b are non-negative.

Otherwise, if I multiply this with a negative number, I will get minus 3 X 1 minus 3 X 2 will be less than or equal to minus 63 or if I multiply with minus 1, I will get minus 3 X 1 minus 3 X 2 is less than equal to minus 21, which will become 3 X 1 plus 3 X 2 greater than or equal to 21. So, I should not multiply at the moment, I should not multiply with the negative number. So, a and b have to be greater than or equal to 0.

o, when I multiply the first constraint by a and I multiply the second constraint by b, the new constraint for X 1 will have 3 a plus 4 b. Now, a and b have to be such that, 3 a plus 4 b is greater than or equal to 10 and they also have to be such that, 3 a plus 3 b greater than or equal to 9, so 3 a plus 3 b is greater than or equal to 9. If I am able to get a and b that satisfy these two conditions after multiplication, then 21 a plus 24 b is an upper estimate of the objective function value.

Now, that is the generalization that we have made based on what we have seen earlier. Now, we go back and generalize it further. Now, if we want to minimize that upper estimate, then what are we trying to do. If we want to keep that upper estimate as small as possible, now we have to find a and b such that, 3 a plus 4 b is greater than or equal to 10, 3 a plus 3 b is greater than or equal to 9 and 21 a plus 24 b is as small as possible.

So, now, a and b have to be found such that we minimize 21 a plus 24 b subject to 3 a plus 4 b is greater than or equal to 10 and 3 a plus 3 b is greater than or equal to 9, a and b greater than or equal to 0. Now, we realize that in the process what we have done is, we have formulated another linear programming problem which is this, where now a and b are variables. Now, to give a formal structure to such a linear programming problem, we defined replace a and b by new variables called Y 1 and Y 2.

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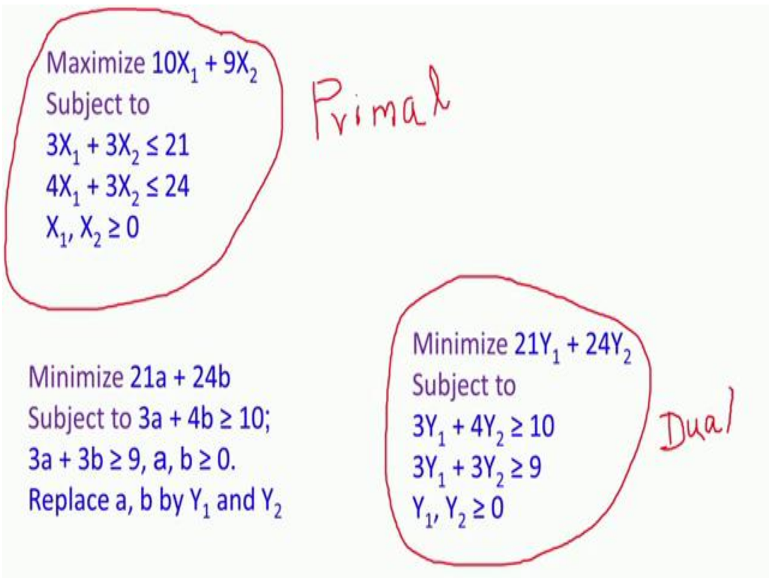

And if we do so, the new problem becomes minimize 21. After doing this, we will get minimize 21 Y 1 plus 24 Y 2 subject to 3 Y 1. Now, a is replaced by Y 1 and b is replaced by Y 2, now 3 Y 1 plus 4 Y 2 is greater than or equal to 10, 3 Y 1 plus 3 Y 2 is greater than or equal to 9, Y 1 Y 2 greater than or equal to 0. So, starting from the given problem which is here we tried to get an upper estimate of the objective function value and we ended up creating another linear programming problem which is given here.

Now, the original problem that we had is called the primal, which is given by p and the new problem is called the dual, it is called the dual. Now, for every linear programming problem which is called as primal, there is another linear programming problem which is called the dual. In the next class, we will see how we can write the dual directly from the primal without going through the process of evaluating the upper estimate.