**Introduction to Operations Research**
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**Module - 03**
**Simplex Algorithm**
**Lecture – 15**
**Infeasibility**

In this class, we will see one more aspect in the simplex table. We consider a specific linear programming problem, solve it using the Simplex Algorithm and the table. And then when we find something specific happening, we will try to explain what happens to the linear programming problem.

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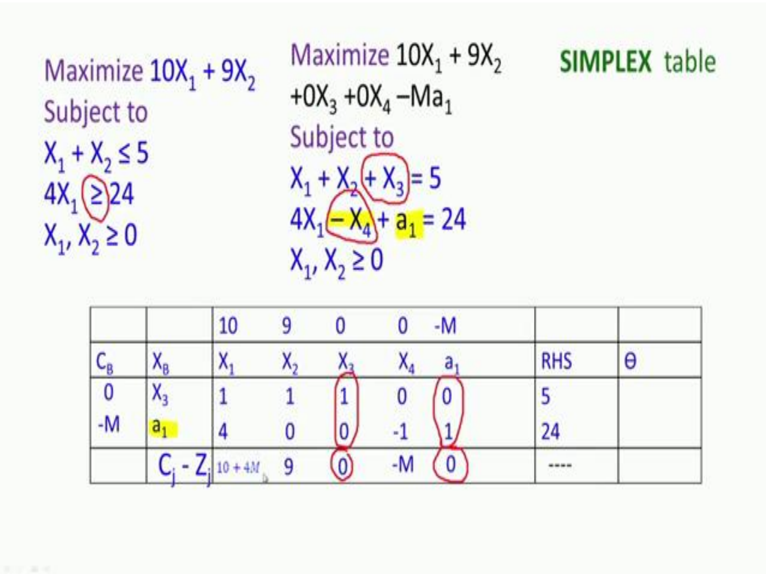


So, let us consider this linear programming problem with 2 variables and 2 constrains, it is a maximization problem with 10 X 1 plus 9 X 2 familiar objective function, but the constraints are different. We have X 1 plus X 2 less than or equal to 5 and 4 X 1 greater than or equal to 24 with X 1 X 2 greater than or equal to 0.So, as usual we convert these inequalities to equations by adding slack variables. Now, X 1 plus X 2 less than or equal to 5 is converted to an equation by adding a positive slack variable X 3.

So, plus X 3 equal to 5, X 3 greater than or equal to 0, now 4 X 1 greater than or equal to 24 will involve a negative slack. So, 4 X 1 minus X 4 is equal to 24 and X 1, X 2, X 3, X

4 greater than or equal to 0. So, as we have seen before, this greater than or equal to constraint has resulted in a negative slack variable and this negative slack variable cannot be a starting variable for this simplex algorithm, therefore we introduce the artificial variable here with a 1.

So, a 1 is the artificial variable, so X 1, X 2, X 3, X 4 and a 1 greater than or equal to 0. The two slack variables X 3 and X 4, one is a positive slack and the other is a negative slack, they do not contribute to the objective function. The a 1 variable to the minimization problem would have contributed a big M and for the maximization problem, it contributes a minus big M. The motivation again as we saw earlier is that, now we have written a new problem by introducing the artificial variable.

If this problem is going to be solved from which we are going to get the solution the original problem, then we have to make sure that the a 1 does not come in the solution of this problem. The way by which we implemented is, for a maximization problem if the co efficient is a very, very small number. Remember, M is a large positive number tending to infinity, minus M is negative of M therefore, it is a large number, but with a negative sign and therefore, it is not expected to appear in the solution, because it is contribution is very, very small.

In fact, it is contribution is negative, so it is not expected to come into the solution of maximization. So, this is the new problem that we will solve, because of the greater than or equal to constraint which necessitated in a negative slack and necessitated in a artificial variable. There is only one artificial variable, there are no two artificial variables, there is only one and that has a minus M, so we now set up the simplex table to solve this.

So, there are five variables, the two decision variables X 1, X 2, the two slack variables X 3, X 4 and the artificial variable a 1 that we have here. So, we write the objective function values 10, 9, 0, 0 and minus M that comes. Please note that this minus M is coming, because of the artificial variable and the maximization function, so there is a minus M in the objective here. The equations are X 1 plus X 2 plus X 3 plus 0 X 4 plus 0 a 1 equal to 5.

So, we write X 1 plus X 2 plus X 3 plus 0 X 4 plus 0 a 1 is equal to 5. Second equation is 4 X 1 minus X 4 plus a 1, so 4 X 1 plus 0 X 2 plus 0 X 3 minus X 4 plus a 1. So, 4 X 1 plus 0 X 2 plus 0 X 3 minus X 4 plus a 1 equals 24, so we write the two basic variables. Now, we realize that, they can start the simplex with the X 3 and a 1. Now, X 3 has… X 3 and a 1 put together actually have the identity matrix. In this particular example, X 2 and a 1 also can give us the identity matrix, but we start with X 3 and a 1 to be consistent, that if there is a slack variable we start with the slack variable.

So, X 3 and a 1 are the two variables with which we start the solution. Now, the objective function coefficient of these two variables are X 3 has a 0 contribution, a 1 has a minus M contribution, so 0 and minus M. Now, we need to check whether this solution is optimal or whether there is a variable that can enter the solution with the positive C j minus Z j, so we calculate this C j minus Z j values. Now, for variable X 1, it is 0 into 1 minus M into 4 therefore, it is minus 4 M.

So, 10 minus, minus 4 M is 4 M plus 10 or 10 plus 4 M which is shown here, 10 plus 4 M. For the variable X 2, it is 0 into 1 minus M into 0 is 0, so 9 minus 0 is 9, so this value is 0 into 1 minus M into 0 0, so 9 minus 0 is 9. For variable X 3, the Z j value is 0 into 1 minus M into 0 0, so 0 minus 0 will be 0. For variable X 4, it is 0 into 0 plus minus M into minus 1 is plus M, so 0 minus M is minus M and for the variable a 1, it is 0 into 0, 0 minus M into 1 is minus M, minus M minus, minus M will be 0.

Once again you can observe that, the two basic variables X 3 and a 1 will have C j minus Z j values as 0, they will have C j minus Z j values as 0. The value of the objective function will be 0 into 5 plus minus M into 24, but since an artificial variable is in the solution. We do not write the value of the objective function, instead we keep this value as dash. So, we now have to check whether there is a variable that can enter the solution.

So, X 2 can enter the solution with 9, X 1 can enter the solution with 4 M plus 10, now M is large and positive therefore, 4 M plus 10 is bigger than 9. Therefore, the variable X 1 will enter the solution, because it has the larger value of C j minus Z j between the two variables. Once again this is 9, this is 4 M plus 10, M is large and positive and tends to infinity therefore, 4 M plus 10 is bigger than 9, so variable X 1 enters the solution.

Now, X 1 has to replace either X 3 or a 1, so we find the ratios. 5 divided by 1 is 5 and 24 divided by 4 is 6, the smaller one is 5, now X 1 will come into the solution and replace X 3 and the artificial variable stays in the solution. Now, let us look at the next iteration.

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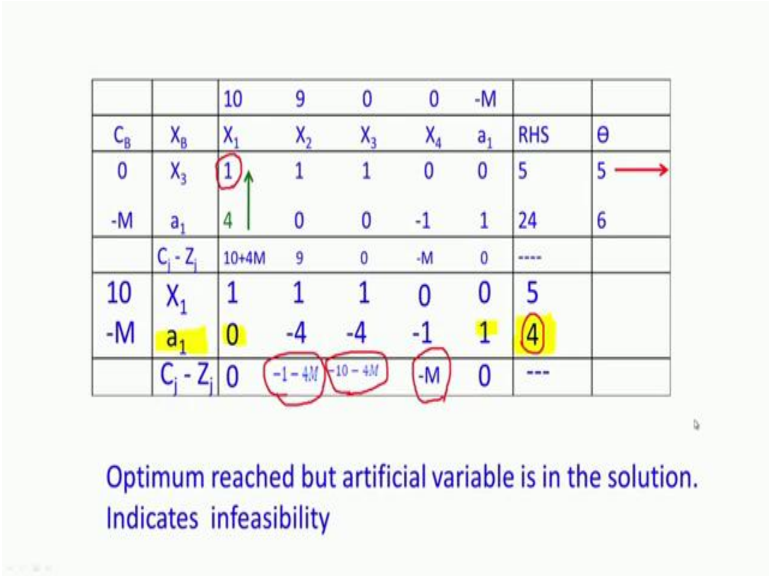

| | | 10 | 9 | 0 | 0 | -M | | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | $X_B$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $a_1$ | RHS | θ |
| 0 | $X_3$ | 1 | 1 | 1 | 0 | 0 | 5 | 5 → |
| -M | $a_1$ | 4 | 0 | 0 | -1 | 1 | 24 | 6 |
| | $C_j - Z_j$ | 10+4M | 9 | 0 | -M | 0 | ---- | |
| 10 | $X_1$ | 1 | 1 | 1 | 0 | 0 | 5 | |
| -M | $a_1$ | 0 | -4 | -4 | -1 | 1 | 4 | |
| | $C_j - Z_j$ | 0 | -1 − 4M | 10 − 4M | -M | 0 | --- | |



So, I capture the first iteration in this table, where variable X 1 enter the solution with 4 M plus 10 and the variable X 3 left the solution. Now, this is your pivot element, now the new basic variables are X 1 replaces X 3 and a 1 remains, so we will have X 1 and a 1 in the solution. X 1 is replacing X 3, so X 1 comes here, a 1 remains. The value of the objective function coefficients are 10 and minus M, X 1 has 10, a 1 has minus M.

Now, this is the pivot element, so divide every element of the pivot row by the pivot element, since the pivot element happens to be 1, the same values will repeat. So, 1 by 1 is 1, again 1 by 1 is 1, again 1 by 1 is 1, 0 divided by 1 is 0, 0 divided by 1 is 0 and 5 divided by 1 is 5. Now, variable X 1 is in the solution with the 1, so I need to get a 0 in this position, as well as retain a 1 in this position, retain a 1 corresponding to a 1, we need to do that.

Now, we go back and visit the corresponding position which is 4, so 4 I need a 0, 4 minus 4 times 1 is 0. So, every element here I take this, and then I multiply this 1 by 4 and subtract. So, I will have 4 minus 4 times 1 is 0, 0 minus 4 into 1 is minus 4, 0 minus 4 into 1 is minus 4, minus 1 plus 0 into 4 is minus 1, minus 1 plus 0 into 4 is minus 1, plus 1 minus 0 into 4 is plus 1 and 24 minus 5 into 4 20 is plus 4. Now, this is the solution X 1 equal to 5, a 1 is equal to 4.

Now, we want to check whether this solution is the best solution. To do that, we have to find out whether there is a variable that can first come in to the solution, for which we have to find out the values of C j minus Z j. So, we write C j minus Z j and find out the values. 10 into 1 minus M into 0 is 10, 10 minus 10 will be 0, 10 into 1 is 10, minus M into minus 4 is 4 M. So, 9 minus 10 minus 4 M is minus 4 M minus 1.

For variable X 3, 10 into 1 is 10, minus M into minus 4 is 4 M, so this part is 4 M plus 10, 0 minus 4 M plus 10 is minus 4 M minus 10, written as minus 10 minus 4 M. For variable X 4, it is 10 into 0 0, minus M into minus 1 is plus M, so 0 minus plus M is minus M and for variable a 1, 10 into 0 is 0, minus M into 1 is minus M, so this minus M minus, minus M will be 0. Now, the value of the objective function is 10 into 5 50 plus minus M into 4, so 50 minus 4 M, but we do not write it, because the artificial variable is in the solution. Therefore, we write only a dash coming here.

Now, we want to check whether this solution is optimum, we have reach the optimum solution or is there a variable with a positive value of C j minus Z j, that can come into the solution. Now, variable X 1 and a 1 are basic, so we do not consider them, we consider X 2, X 3 and X 4. Now, we realize that this value of minus 1 minus 4 M is negative, because M is large and positive, so minus 4 M is negative, so minus 4 M minus 1 is negative.

Similarly, minus 10 minus 4 M is also negative, because M is large and positive. So, minus 4 M minus 10 is also negative, minus M is also negative. Therefore, there is no entering variable, the algorithm terminates normally without any difficulty. The algorithm terminates and we could say that X 1 equal to 5, a 1 equal to 4 is the solution.

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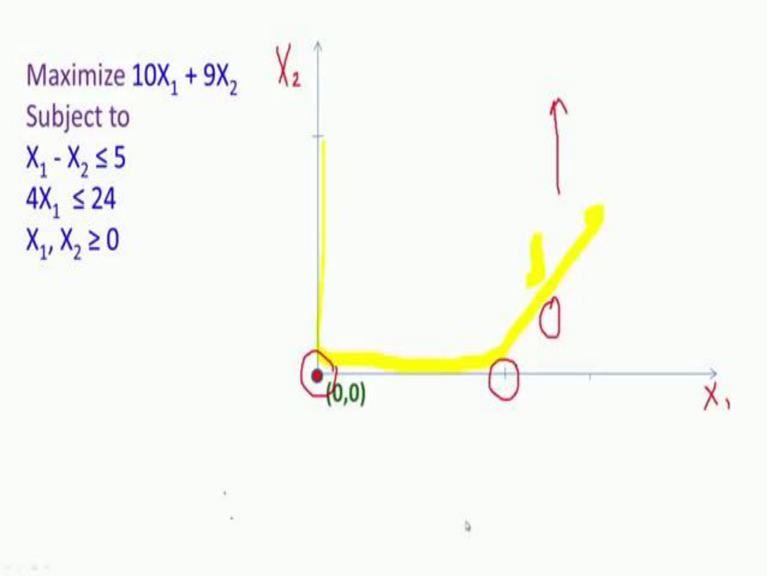


But, when we started with the simplex algorithm ((Refer Time: 15:19)) for this particular problem, we wrote the artificial variable here, because this constraint had greater than or equal to. We wrote an artificial variable here and this artificial variable came into the solution. Now, when wrote this artificial variable, this artificial variable was introduced only for the sake of starting the simplex algorithm and this was not part of the original problem.

So, we said we do not want this artificial variable to be in the solution and the way we handled it is by giving a large positive value M and we are putting a minus M in the maximization objective, so that a large negative contribution prevents this variable from being in the solution. So, we thought that, by putting a large negative contribution here, big M is large and positive, so minus M is a large negative. So, by putting a large negative contribution we will prevent this a 1 from coming into the solution.

So, that if this problem has a solution, now this problem will show the same solution and this problem solution will not involve a 1.

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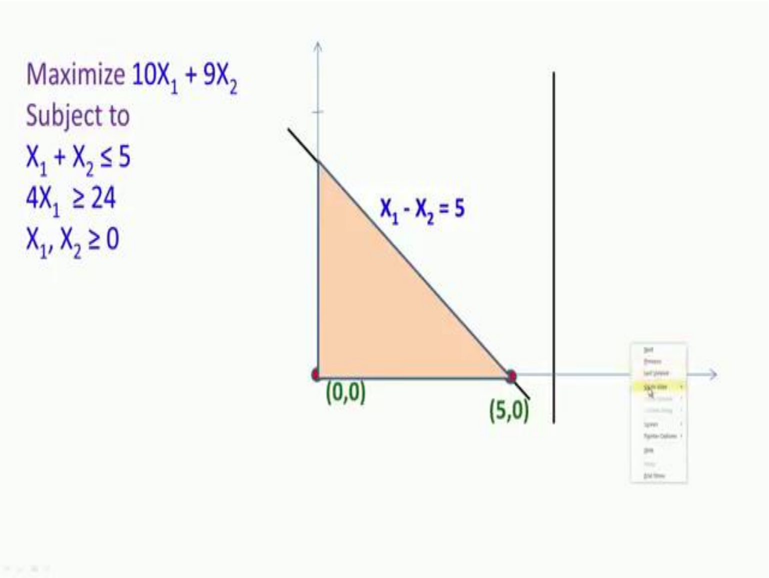


So, we started with that principle, but we observe that, when simplex terminated a 1 is actually in the solution and we do not want it. So, we are in spite of us putting a big M and putting a large negative contribution, thinking that the variable a 1 will not be in the solution. We now observe that the variable ((Refer Time: 17:23)) a 1 is in the solution, ((Refer Time: 17:34)) so the variable a 1 is in the solution.

So, once again revisiting this, we said that if this has a solution, then by putting a large negative value we will prevent a 1 from coming into the solution and solving this problem will give the same solution to that of this problem. But, now in spite of us putting a negative contribution, a 1 is in the solution. It means that the original problem does not have a solution at all, now this is called in feasibility. The original problem does not have a solution.

So, infeasibility is indicated by ((Refer Time: 18:17)) simplex terminating normally, but after simplex terminates there is an artificial variable in the solution with a strictly positive value. There is an artificial variable with a strictly positive value when simplex terminates and there is an artificial variable in the solution with strictly positive value we say that it is invisible. Now, we will also note this value, this value is 4, and then we will see how the graphical method looks at the same problem.

So, we now go back to the ((Refer Time: 19:03)) optimum is reached, but artificial variable is in the solution, indicates infeasibility of the linear programming problem.

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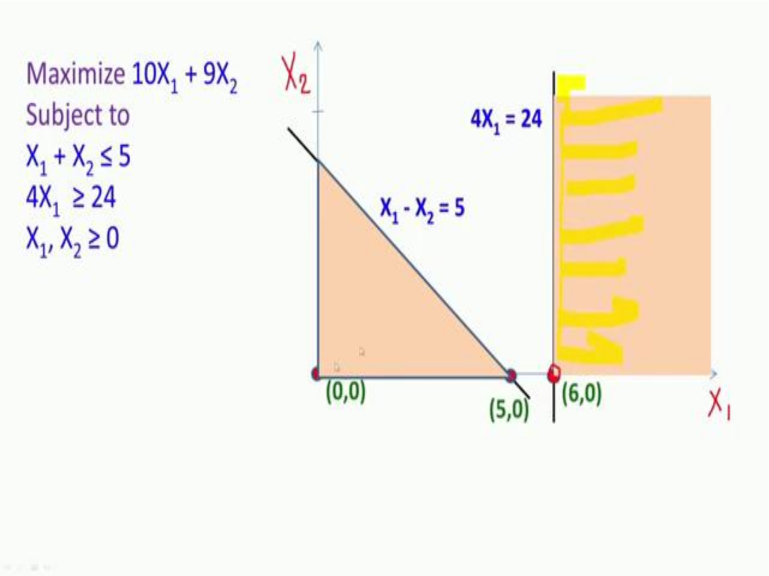


Let us go to the graphical method, we have the same problem, we draw the axis with 0 and 0, now this is for variable X 1 and this is for variable X 2. We now draw the two lines X 1 plus X 2 equal to 5 is drawn here with points 5 0 and 0 5. This is a point 0 5, now origin 0 0 satisfies the inequality, X 1 plus X 2 is equal to 5. So, this is the feasible region, the triangle that we are showing in different color is the feasible region to this problem.

Every point on this including these points is the feasible region for the first constraint. Now, we draw the second constraint, 4 X 1 equal to 24 or X 1 equal to 6. Now, X 1 equal to 6 is a line parallel to the X 2 axis or the Y axis passing through the point X 1 equal to 6 and this point is 6 comma 0. So, this point will be 6 comma 0, so this is 6 comma 0 and this is a line for X 1 equal to 24. Now, we have to find out that region which satisfies the inequality, so the origin is here 0 0 is actually less than 24.

Therefore the other region is the region that satisfies this inequality, this is the other region is the region that satisfies the inequality. So, we show the other region, now this way. Now, we realize that the other region is the region that satisfies the inequality. Now, if we look at, the feasible region associated with the two constraints, we have one constraint where this portion is the feasible region. For one constraint, we have this portion has feasible region and for the other constraint, the feasible region is somewhere here.

So, we realize that this feasible region there is no actual region which satisfies both these constraints. When we have two constraints, where there is no set of point, where there is no point satisfying this together, then we say that the problem is infeasible. Now, this infeasibility is indicated by the simplex algorithm with the presence of an artificial variable in the solution. Now, that artificial variable had a value a 1 is equal to 4, now that a 1 was actually introduced in this constraint, in the second constraint 4 X 1 greater than or equal to 24.

So, this constraint became 4 X 1 minus X 4 plus a 1 equal to 24. So, if a 1 is equal to 4, we would get 4 X 1 greater than or equal to 20, which is X 1 greater than or equal to 5 and if we have plotted that, now this line 4 X 1 equal to 24 would have become 4 X 1 equal to 24 and it would have come somewhere here, showing that the point 5 comma 0 is the optimum solution. But, because we have 24, the LP solution was suggesting that a 1 should be equal to 4, so that 4 X 1 becomes greater than or equal to 20 and we can get a solution.

But, right now there is no solution, because a 1 is equal to 4. So, the presence of the artificial variable in the solution not only indicates infeasibility, but also gives us a way to understand the amount of infeasibility or the extent of infeasibility for the linear programming problem. So, at the end we say that there is a no region that satisfies both the constraints, the given linear programming problem is infeasible.

So, in this class we have seen feasibility, in the earlier class we saw unboundedness and we saw, how simplex handles both infeasibility and unboundedness. So, there are essentially three things to a linear programming problem. The linear programming problem, any problem either has an optimum or is bounded or is infeasible. Only one out of the three can actually happen for a given linear programming problem.

We have right now seen all the three instances; there are still a few more things that can happen when we actually do this simplex iteration. Those things we will study through examples and exercise as we move along in this course. In the module, we will move forward to study another interesting aspect of linear programming which is called duality.