Introduction to Operations Research
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Module - 03
Simplex Algorithm
Lecture - 04
Unboundedness

In this class, we consider some more aspects of the Simplex Algorithm. We now try to solve a specific linear programming problem by the simplex algorithm and try to understand, how simplex handles certain situations relevant to the linear programming problem.

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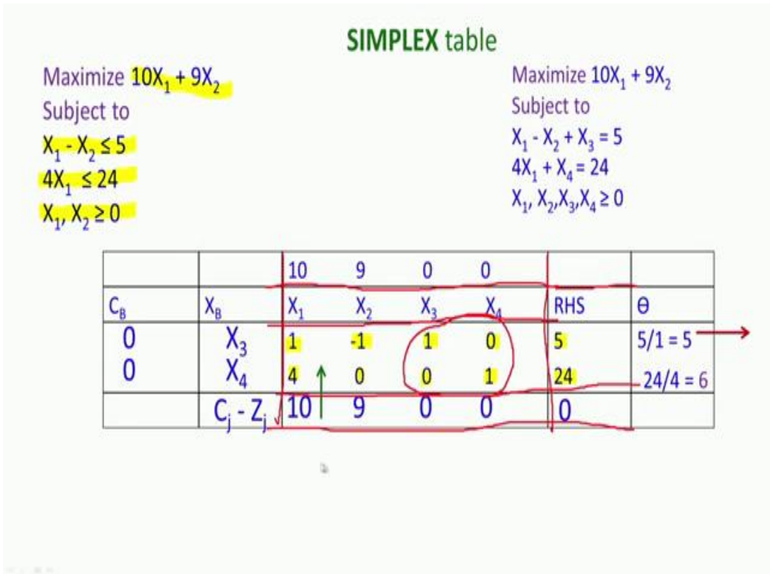

So, let us look at this problem which is given here, maximize 10 X 1 plus 9 X 2, subject to X 1 minus X 2 less than or equal to 5 and 4 X 1 less than or equal to 24, X 1, X 2 greater than or equal to 0. So, we will try to solve this linear programming problem using the simplex algorithm, using the things that we have already seen in the earlier classes. Now, we first add slack variables to convert these inequalities to equations, we add X 3 and X 4, so X 1 minus X 2 plus X 3 equal to 5, 4 X 1 plus X 4 equals 24 and we maximize 10 X 1 plus 9 X 2 plus 0 X 3 plus 0 X 4.

So, we have four variables that we have, so we setup the simplex table. So, we setup this simplex table this way, we first draw these two lines and then, we draw this line and then, we can draw this line. So, we write X 1, X 2, X 3, X 4 here, we write the objective function coefficients 10, 9, 0 and 0 above the four variables. We then draw this line like this, we also write X 1, X 2, X 3, X 4 and right hand side here.

We write the equations as they are X 1 minus X 2, X 1 minus X 2 plus X 3 equal to 5. So, this is written as X 1 minus X 2 plus X 3 plus 0 X 4 equal to 5, 4 X 1 plus X 4 equals 24 is written as 4 X 1 plus 0 X 2 plus 0 X 3 plus 1 X 4 equal to 24. We now have to write the basic variables. Now, X 3 and X 4 automatically become the starting variables, because these are inequalities with less than or equal to sign.

So, slack variables X 3 and X 4 are added, they will automatically become the initial set of variables and we can also see that corresponding to X 3 and X 4, we have the identity matrix, which is here 1 0, 0 1. So, X 3 and X 4 are automatically become the starting variables. We now write the objective function coefficients, 0 0 are written here and we then draw this line and then, calculate the C j minus Z j.

So, C j minus Z j for variable X 1, which is called C 1 minus Z 1 will be 0 into 1 plus 0 into 4 is 0, 10 minus 0 is 10. Similarly, 0 into minus 1 plus 0 into 0, so 0, 9 minus 0 is 9, X 3 and X 4 will have 0 value, because they are basic variables. The value of the objective function at X 3 equal to 5, X 4 equal to 24 is 0. Now, we need to check whether the optimum solution has been reached, now we look at the C j minus Z j values. There is a variable with a positive value of C j minus Z j.

In fact, there are two variables; we choose the one that has the more positive value of the two. So, we choose variable X 1 can come into the solution and improve the objective function from 0. Therefore, this solution is not optimal, so X 1 can come into the solution. Now, X 1 has to replace either X 3 or X 4, so we calculate 5 divided by 1, this is the entering variable, this is called the entering column.

So, the right hand side divided by the coefficient in the entering column 5 divided by 1 5, 24 divided by 4 6. So, the limiting value is 5, the smaller value. So, variable X 3 will leave the solution, so variable X 1 will replace variable X 3 in the next iteration.

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		10	9	0	0		
C_B	X_B	X_1	X_2	X_3	X_4	RHS	θ
0	X_3	1	-1	1	0	5	5
0	X_4	4	0	0	1	24	6
	$C_j - Z_j$	10	9	0	0	0	
10	X_1	1	-1	1	0	5	
0	X_4	0	4	-4	1	4	1
	$C_j - Z_j$	0	19	-10	0	50	
10	X_1	1	0	0	¼	6	X
9	X_2	0	1	-1	¼	1	X
	$C_j - Z_j$	0	0	9	-19/4	69	

No leaving variable. Algorithm terminates showing unboundedness

So, we move to the next iteration. So, I have shown the same part of the simplex table with variable X 1 coming into the solution and variable X 3 leaving the solution. So, the new set of basic variables are, X 1 and X 4, because X 1 replaces X 3. So, we write the new set of basic variables, we simply extend these two lines. So, we just extend these two lines and then, we can draw these lines as well.

So, we write the two basic variables that are X 1 and X 4, the variable X 1 has replaced X 3. So, we have X 1 and X 4. The objective function coefficients are 10 and 0, 10 for variable X 1 and 0 for variable X 4. Now, this is our pivot element, this is the leaving variable, so this is our pivot element. So, we write the first row again, so divide every element of the pivot by the pivot element. So, 1 divided by 1 is 1, minus 1 divided by 1 is minus 1, 1 divided by 1 is 1, 0 divided by 1 is 0, 5 divided by 1 is 5, we get the same row repeating again.

Now, that variable X 1 has come into the solution, I need a 0 here, because X 1 and X 4 should have the identity matrix of 1 0 and 0 1. So, I need a 0 here, now I look at the value here, which happens to be 4, so 4 minus 4 times 1 is 0, this is the pivot row of the new table. So, this will be 1, so 4 minus 4 times 1 is 0. So, we subtract from every element here, this element multiplied by 4, so 4 minus 4 times 1 is 0. So, 4 minus 4 times 1 is 0, 0 minus 4 into minus 1 is plus 4, 0 minus 4 into plus 1 is minus 4 and 1 minus 4 into 0 is 1, 24 minus 5 into 4 20 is equal to 4.

Now, we have to check whether we have got the optimum solution here. To do that, we have to compute the values of C j minus Z j, so we compute C j minus Z j. Now, C 1 minus Z 1, 10 into 1 plus 0 into 0 is 10, 10 minus 10 is 0, also X 1 is a variable in the solution, so it is a basic variable. So, it is C j minus Z j will become 0. Now, for variable X 2, 10 into minus 1 is minus 10 plus 0 into 4 0 is minus 10, 9 minus, minus 10 is 19.

For variable X 3, 10 into 1 is 10 plus 0 into minus 4 is 0, 0 minus 10 is minus 10 and for variable X 4, 10 into 0 plus 0 into 1 is 0, 0 minus 0 is 0. Also observe that X 4 is a basic variable, it is C j minus Z j value will become 0. Now, the value of the right hand side is 10 into 5 plus 0 into 4, which is 50. Now, again we need to check whether this solution is optimum or the best solution.

So, we are able to find one variable X 2, which is non basic, which is not in the solution, it has a positive value of C j minus Z j. So, variable X 2 can come into the solution. So, X 2 comes into the solution and X 2 can replace either X 1 or X 4. Now, we need to see which one it replaces. So, we find the ratios, 5 divided by minus 1, I have already mentioned once that, if the denominator is negative or 0, we do not do the calculation.

Therefore, we do not compute the ratio here; you do not compute the ratio here, because 5 divided by minus 1 will not give a positive value or a 0 here. So, we will not calculate. So, 4 divided by 4 is 1, there is only one value of the ratio. So, automatically that will be minimum, so variable X 4 will leave the solution and variable X 2 will come into the solution. We now proceed to the next iteration of the simplex algorithm; I have shown the two previous iterations already here.

We do not require this arrow; this is the arrow which says that variable X 4 comes into the solution. So, we now write the two variables, variable X 2 has replaced X 4, so X 1 and X 2 are the variables in the solution. Now, the values of the objective functions are 10 and 9, so we write the values of the objective function that are 10 and 9. Now, we observe that from this, this is the entering variable, this is the leaving variable, the arrows shown slightly below, but this is the leaving variable. So, this is our pivot element.

So, now we have to first fill the column corresponding to X 2, where we divide every element of the pivot row by the pivot element. So, we will have 0 divided by 4, which is 0, 4 divided by 4, which is 1, minus 4 divided by 4, which is minus 1, 1 divided by 4, which is 1 by 4 and 4 divided by 4, which is 1. Now, we have to fill the next row and to do that, we realize, this is the variable that has come in, so I need a 0 in this position. Since, I need a 0 in this position, so that I get the identity matrix.

Now, I look at the previous position which is minus 1. So, minus 1 plus 1 will give me a 0. So, what I do is, I add every element of this row to the corresponding element here and fill the new row for X 1. So, 1 plus 0 is 1, minus 1 plus 1 becomes 0, 1 minus 1 becomes 0, 0 plus 1 by 4 becomes 1 by 4, 5 plus 1 becomes 6. Now, we need to check whether this solution X 1 equal to 6, X 2 equal to 1 is optimum, to do that, we find out the C j minus Z j values.

So, we find C j minus Z j for the first variable 10 into 1 plus 9 into 0 is 10, 10 minus 10 is 0. For the second variable, 10 into 0 plus 9 into 1 is 9, 9 minus 9 0, once again we observe that the two basic variables or variables in the solution have 0 values for C j minus Z j. For the third variable, 10 into 0 plus 9 into minus 1 is minus 9, 0 minus, minus 9 is plus 9 and for the 4th variable 10 into 1 by 4 plus 9 into 1 by 4 is 19 by 4, 0 minus 19 by 4 is minus 19 by 4. The value of the objective function is 10 into 6, 60 plus 9 into 1, 9 gives us 69.

There is also an interesting thing, which I have so far not mentioned to you, which I would do right now. Now, if we look at this table in the first iteration, the value of the objective function was 0, X 1 with the C j minus Z j value of 10 was entering and the corresponding value for the leaving variable was 6, so 10 into 6 is 60. So, we add that to the old solution of 0, 5 was leaving, 10 into 5, 5 was leaving. So, 10 into 5 is 50, so 50 plus 0 is the value here, which is 50.

Now, here the object function value is 50, X 2 with 19 is coming into the solution, X 4 with the ratio of one was leaving. So, 19 plus into 1 is 19, 50 plus 19 is 60 9. So, the objective function improves in every iteration by the product of the entering C j minus Z j and the corresponding ratio associated with the leaving variable, 10 into 5, 50, so 0 became 50, 19 into 1, 19, 50 became 69. So, right now the solution is X 1 equal to 6, X 2 equal to 1 with objective function value 69.

Now, we want to check whether this solution is the best solution or whether this solution is the optimum solution. So, we check whether there is a non-basic variable which has a positive C j minus Z j. Now, variable X 3 has a positive C j minus Z j with the value of 9 and variable X 3 can come into the solution. Now, you will observe something interesting, we started with X 3 and X 4, now here X 3 left the solution. So, it became X 1 and X 4, now X 3 is trying to come in to the solution.

So, you may have a question can a variable that has left the solution, can it come into the solution again, the answer is yes, it can try to coming into the solution, nothing prevents it from coming. The only change is you will realize that, when it comes or at any iteration irrespective of whether an earlier variable that had earlier left is coming in or not, one thing that you have to see is this combination, this set of basic variables, which is called bases will be different in different iterations.

Here, it is X 3, X 4; here it is X 1, X 4; here it is X 1, X 2 and so on and the bases will differ with one variable new coming in replacing one of the variables in the existing bases. Therefore, we need not be worried about the fact that X 3 is trying to come into the solution. So, when X 3 is trying to come into the solution, X 3 has to replace either X 1 or X 2, which means, we have to find a ratio.

So, we try to find 6 divided by 0, 6 divided by 0, we will not do, because the denominator is 0. Therefore, we will not calculate this ratio. The next one will be 1 divided by minus 1. So, once again the denominator is negative, therefore, we will not calculate this ratio. Therefore, we are not able to get a value for this ratio and no variable is willing to leave the solution.

So, we have a peculiar situation, where a variable is trying to come into the solution, but no variable is leaving the solution. Now, this can happen and this has happened to the particular example that we are right now looking at. So, it so happened that this variable is X 3, which had already left, but this phenomenon has nothing to do with the fact that X 3 had already left. It could happen to any variable, you may encounter a situation where this variable can come into the solution, but it so happens that the column associated with this variable does not have any positive number.

So, when it does not have any positive number, there will not be any ratio, the ratios cannot be calculated. So, the algorithm has to terminate, when there is an entering variable and when there is no leaving variable and the algorithm terminates showing, what is called Unboundedness. So, the algorithm terminates with what is called Unboundedness.

So, a simplex now can terminate in a second way, earlier, we said the algorithm stopped or terminated, when there is no entering variable, when there was no variable with the positive C j minus Z j. Now, we can have a situation where there is a variable that can enter, but we are not able to find a leaving variable, therefore, the algorithm terminates indicating Unboundedness. Now, what is Unboundedness? Now, we will see that by drawing the graph corresponding to the same linear programming problem.

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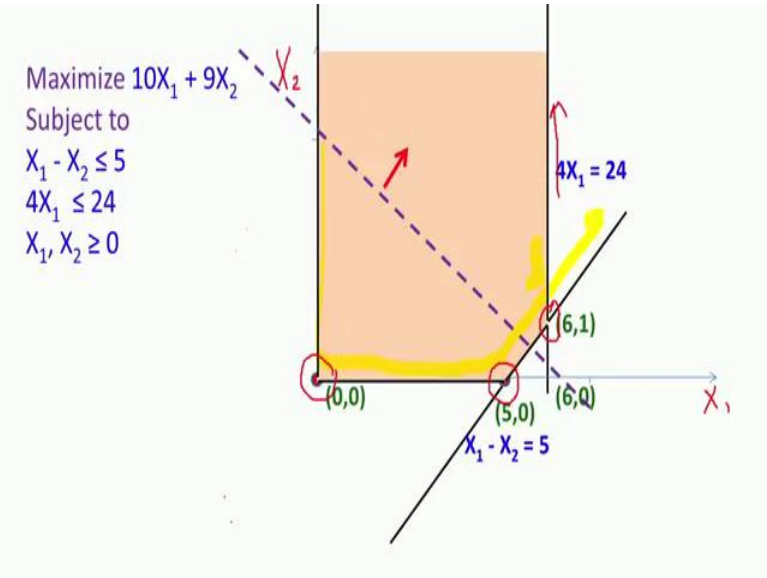

Now, this is the linear programming problem that we are solving. Now, we try to draw the graph for this linear programming problem. So, we first start the graph by drawing the X, the X and Y axis. So, this stands for variable X 1, this stands for variable X 2. So, we have written these two variables. Now, we draw the first line, which is X 1 plus X 2 equal to 5, passing through the point 5 comma 0 and we understand that the origin satisfies, therefore, this area to the left of this, this is the area that satisfies the inequality X 1 plus X 2, this is typically the area that satisfies the inequality.

Now, we now move on to draw the second line, which is X 2 for X 1 equal to 24 or X 1 equal to 6 is a line parallel to the Y axis passing through 6 comma 0. Once, again the origin is here satisfying, therefore, the region that will satisfy is this region, the left region, this is the region. So, we now show the feasible region or the region that satisfies this.

Now, we realize that this region is unbounded in the sense that, there is there is no finite region that we have. There are three distinct corner points, but there is no finite region that we have. So, if we draw the objective function 10 X 1 plus 9 X 2 and then, if we mark the direction of movement of the objective function, we draw iso parallel lines and try to move this in the direction of the objective function.

We will now realize that this objective function line will move like this will keep moving this way. And finally, it will leave you will have these iso objective function lines it will finally, come here touch and it will keep going, because there is no limit to the feasible region and it will never leave the feasible region. So, we observe that, it will never leave the feasible region.

So, if we move the objective function in the direction of movement, the objective function will move upwards and keep going upwards and will never the feasible region. So, this phenomenon is called Unboundedness. So, when the objective function does not leave the feasible region, we say that the linear programming problem is unbounded. So, let us map the simplex algorithm with this.

Now, the simplex algorithm went through three iterations, it started with the solution 0 0, which we had here, then it move to the solution 5 comma 0 with the objective function 50. Then, it move to the solution 6 comma 1 with the objective function 69 and then, it we realize that, it is moving upwards and it is not getting the next point. Therefore, the feasible, there is no optimum solution to this problem, the problem is unbounded, simplex algorithm showed exactly that phenomenon.

So, simplex showed the same phenomenon, it went through three iterations, it first showed the solution with objective function equal to 0. Then, it moved to the solution with objective function 50 and then, it moved to 69 and then, it is unable to find a leaving variable, it terminates showing Unboundedness. So, through this example, we have understood that certain linear programming problems are unbounded and when the linear programming problem is unbounded, simplex will encounter a situation, where there is an entering variable. There is no leaving variable and it will terminate showing Unboundedness.

In fact, if we have went to the very first iteration, instead of entering this 10, though we have consistently entered the one with the largest C j minus Z j. Now, 9 with the positive

C j minus Z j is actually a candidate that can enter. So, if we had chosen to enter this 9 right here, we realize that 5 divided by minus 1 and 24 divided by 0 are not possible and Unboundedness is shown here.

So, either Unboundedness will be shown right at the beginning or if we systematically solve it by simplex at some point, it will indicate Unboundedness. There is one more aspect to the linear programming problem and that we will see in the next class.