Introduction to Operations Research
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Module – 03
Simplex Algorithm
Lecture - 03
Tabular form (Minimization)

In this class, we look at the simplex table for Minimization Problems. We consider the same example that we used in the graphical and in the algebraic method to solve minimization problems.

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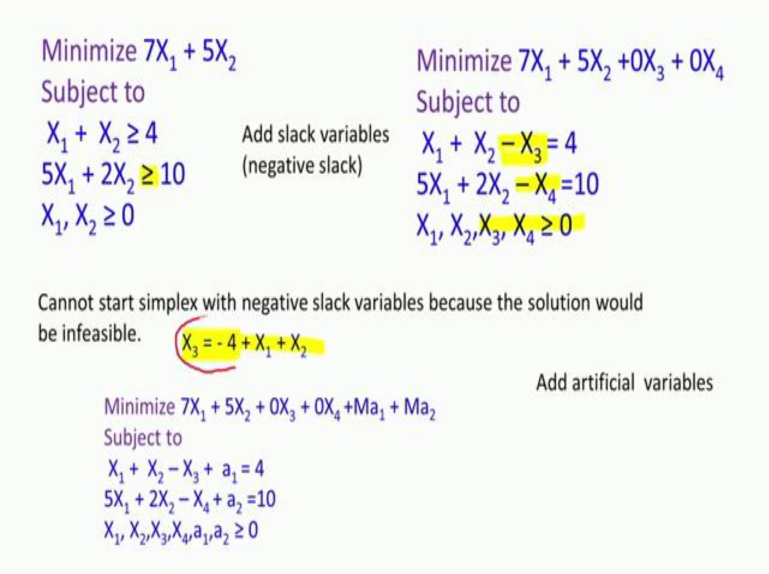

So, minimize 7 X 1 plus 5 X 2, subject to X 1 plus X 2 greater than or equal to 4, 5 X 1 plus 2 X 2 greater than or equal to 10, X 1, X 2 greater than or equal to 0. Now, these inequalities are now converted to equations by adding slack variables, because we have a greater than or equal to constraint. We add a negative slack here minus X 3 with X 3 greater than or equal to 0 and because, we have this greater than or equal to constraint, we add a minus X 4 with X 4 greater than or equal to 0. So, after the addition of the two slack variables, negative slack variables, we get these two equations.

When we solved a maximization problem, we started with the slack variables as the initial basic variables, because we were able to get a solution. Now, we try to start with X 3 and X 4 as the basic variables to begin with and if we take the first constraint or first equation and write in terms of X 3, we realize that if X 1, X 2 are non-basic and they are at 0, we realize that X 3 takes a negative value. We realize that X 3 now takes a negative value, so X 3 becomes equal to minus 4.

So, we will not be able to start the solution with X 3 and X 4. Another way of looking at it is, if we started with X 3 and X 4 and if we left out X 1 and X 2, we would have a solution X 3 equal to minus 4, X 4 equal to minus 10 with which we cannot start the simplex table. The simplex table should start with a basic feasible solution. This would give us an infeasible situation.

So, what we do now is, we observed that, if we had a positive slack variable, a slacked variable with a plus 1 coefficient, then we could have started with that. But, the greater than or equal to constraint, does not give us the scope to add a slack variable with a positive coefficient and non-negative. So, we do not have that opportunity and therefore, only for the sake of starting the simplex table, we add two variables.

Now, only for the sake of starting the simplex table, we add these two variables, we put a plus a 1 and a plus a 2 in these two equations. Now, these variables are called artificial variable, because they do not belong to the problem. In a way the slack variables belong to the problem because, if the problem has this inequality and if I am writing this inequality as an equation here, then automatically the slack variable comes. So, it is part of the problem.

Now, these artificial variables are not part of the problem, they are introduced only for the purpose of starting the simplex table. So, we introduced two artificial variables a 1 and a 2 and now, we can start the simplex table with these artificial variables. Now, we know that the slack variables do not contribute to the objective function. Now, we have to find out, what is the contribution of the artificial variable into the objective function.

Now, to understand that, let us go back to the problem without the artificial variable. So, when we look at this problem without the artificial variable, I have a problem here, which is the given problem, which does not have a 1, a 2. Now, I will add a 1, a 2 to this problem and I am going to solve this new problem that has a 1 and a 2. If I have to find the solution to the old problem and the old problem does not have a 1, a 2, then obviously, the solution to this problem should not have a 1, a 2.

But, I will introduce a 1, a 2 and therefore, it is my responsibility to make sure that, when I solve this problem, the solution, if it exists does not have a 1, a 2. How do I do that? What I do is, I give a large objective function coefficient to a 1 and a 2, this big M is large. This big M is like, if I look at the other values like 7, 5, 0 and 0, big M is large, big M is 100, big M is 1000, big M is 10,000, it is a very large value. So, big M is used as a notation to represent a very large positive value that tends to infinity, very large value.

So, if I put a very large coefficient for a 1 and a 2 and I am minimizing, it is very likely that a 1 and a 2 will not come in the solution. There will always, if this problem has a solution, then the objective function value of that will be smaller compare to a solution that would involve a 1 and a 2, therefore it is likely that a 1 and a 2 do not come into the solution. So, I have introduced a 1 and a 2 for the sake of starting the simplex algorithm, at the same time, I am giving a very large positive value to the objective function. So, that in the solution a 1 and a 2 do not come.

So, if the original problem has a solution, the same solution will come and solutions involving a 1 and a 2 will not come, because they would have higher objective function values and I am trying to minimize.

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Tabular form of the **SIMPLEX** algorithm (Minimization)

		-7	-5	0	0	-M	-M		
C_B	X_B	X_1	X_2	X_3	X_4	a_1	a_2	RHS	θ
-M	a_1	1	1	-1	0	1	0	4	4/1 = 4
-M	a_2	5	2	0	-1	0	1	10	10/5 = 2
	$C_j - Z_j$	6M-7	3M-5	-M	-M	0	0	----	

Minimize $7X_1 + 5X_2 + 0X_3 + 0X_4 + Ma_1 + Ma_2$

Subject to

$X_1 + X_2 - X_3 + a_1 = 4$

$5X_1 + 2X_2 - X_4 + a_2 = 10$

$X_1, X_2, X_3, X_4, a_1, a_2 \geq 0$

So, with this let us go and setup the simplex table. So, we setup the simplex table, we look at this problem with the a 1 and a 2 and we setup the table. In the previous example, we have solved it for a maximization problem, now we have a minimization problem.

So, it is easy to write a minimization problem as a maximization problem by multiplying the objective function with a minus 1.

So, I will setup by saying now the variables are here X 1, X 2, X 3, X 4, a 1, a 2; there are six variables, you can see the six variables are here. The objective function coefficients are written with a negative sign, because they are written for a maximization problem. So, this 7 becomes minus 7 minus 5 and so on. Now, I realize that the way I have written this, my identity matrix is coming with a 1 and a 2, because they have been inserted into the problem for a starting solution and therefore, they will have an identity matrix there.

Therefore, I will start the table with a 1 and a 2 as my first set of basic variables. The objective function coefficients are minus M and minus M, which are taken from these two positions. Now, I find C j minus Z j. Now, what is Z 1? Minus M into 1 plus minus M into 5, which is minus 6 M, so C 1 minus Z 1 is minus 7 minus, minus 6 M, which is 6 M minus 7.

Now, you will observe that I am not giving an explicit value to this M, there are two ways of doing it, I can give an explicit value to M, because the other coefficients are smaller like 7 and 5, I could have given M is equal to 100. But, then if there were some other problems, where these coefficients are large, now M should be bigger than that. Therefore, I am simply using M as a number, which is large positive and tends to infinity.

So, I am writing it at 6 M minus 7, if M is 100, then this will become 593, but I am not writing an explicit number, I am still writing big M as a notation to represent a large number. Now, C 2 minus Z 2 is minus M into 1 plus minus M into 2 is minus 3 M. So, minus 5 minus, minus 3 M is 3 M minus 5, minus M into minus 1 plus minus M into 0 is plus M.

So, 0 minus M is minus M, minus M into 0 plus minus M into minus 1 is plus M, 0 minus M is minus M, minus M into 1 plus minus M into 0 is minus M, minus M, minus, minus M is 0 and similarly, it is 0. The basic variables will always have 0 value for C j minus Z j. Now, the value of the objective function is minus M into 4 plus minus M into 10, which is minus 14 M. But, it is customary that you do not write it, you simply put a dash here, when you have these artificial variables in the solution.

After all we do not want them and therefore, when they are there we do not write the objective function value, we simply keep it as dash. So, this completes part of the first one, we now have to find out, whether this there is another variable that can come into the solution. So, now, we look at the C j minus Z j values. So, we find that variable, which has the largest positive C j minus Z j.

Now, this is a negative value, this is a negative value, these two are positive values, as M is large, both are positive values, if M is infinity, then both are infinity. But, if M is large and not infinity, 6 M will be bigger than 3 M and 6 M minus 7 will be bigger than 3 minus 5, because M is sufficiently large. Therefore, the most positive C j minus Z j happens to be for X 1 and the variable X 1 will come into the solution.

Now, X 1 will come into the solution and it has to replace either a 1 or a 2. To do that, we find the ratios, 4 divided by 1 and 10 divided by 5. So, 4 divided by 1 is 4, 10 divided by 5 is 2 and the limiting value is the minimum of these two ratios. And therefore, this variable a 2 will leave the solution and the variable X 1 will replace the variable a 2 into the solution.

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		-7	-5	0	0	-M	-M		
C_B	X_B	X_1	X_2	X_3	X_4	a_1	a_2	RHS	θ
-M	a_1	1	1	-1	0	1	0	4	4
-M	a_2	5	2	0	-1	0	1	10	2
	$C_j - Z_j$	6M-7	3M-5	-M	-M	0	0	----	
-M	a_1	0	3/5	-1	1/5	1	-1/5	2	10/3
-7	X_1	1	2/5	0	-1/5	0	1/5	2	5
	$C_j - Z_j$	0	$\frac{(3M-11)}{5}$	M	$\frac{(M-7)}{5}$	0	$\frac{(-6M+7)}{5}$	---	
-5	X_2	0	1	-5/3	1/3	5/3	-1/3	10/3	
-7	X_1	1	0	2/3	-1/3	-2/3	1/3	2/3	
	$C_j - Z_j$	0	0	-11/3	-2/3	$\frac{(-3M+11)}{3}$	$\frac{(-3M+2)}{3}$	-64/3	

Now, I have shown the first table, where the variable X 1 is replacing the variable a 2. Now, we have to do the next iteration. So, variable X 1 has replaced variable a 2. So, we write the 2 X 1 has replaced a 2 and a 1, continues to be there. So, X 1 and a 1 are our basic variables. The objective function coefficients are minus M and minus 7 from the first row that we have.

Now, this element, which is the intersection of the entering variable and the leaving variable, which is shown in a different color, this is your pivot element and this row is your pivot row. So, as we did previously, divide every element of the pivot row by the pivot element. So, 5 divided by 5 is 1, remember that this is the second row is the pivot row. So, I write the second row first.

So, 5 divided by 5 is 1, 2 divided by 5 is 2 by 5, 0 divided by 5 is 0, minus 1 divided by 5 is minus 1 by 5, 0 divided by 5 is 0, 1 divided by 5 is 1 by 5 and the right hand side is 10 divided by 5, which is 2. Now, this 2 is the same as this 2, which was also obtained as 10 divided by 5. Now, we have to write the a 1 row, the first row. Now, this is X 1 has become a basic variable, therefore this should have 01,1 is already there, I need a 0 here, I also have to retain the 1 here, because a 1 continues to be a basic variable. So, I need a 0 in this position.

So, I look at the previous 1, which is 1. So, 1 minus 1 will give me 0. So, this number minus this number will give me the value that I should have in this position. So, 1 minus 1 is 0, 1 minus 2 by 5, 3 by 5 minus 1 minus 0 is minus 1, 0 minus, minus 1 by 5 is plus 1 by 5, 1 minus 0 is 1, 0 minus 1 by 5 is minus 1 by 5 and 4 minus 2 is 2. So, we have now completed this and we have to write the C j minus Z j values.

So, we write the C j minus Z j. So, minus M into 0 is plus minus 7 into 1 is plus 7. So, minus M into 0 plus minus 7 into 1 is minus 7. So, minus 7 minus 7 is 0, the basic variable will always get a value 0. Second one we have to write carefully minus M into 3 by 5 minus 3 M by 5 minus 14 by 5. So, minus 5 plus 3 M by 5 plus 17 by 5, which on simplification will give us 3 M minus 11 by 5, minus M into minus 1 plus minus 7 into 0 will give us plus M.

So, 0 minus M is minus M, minus M into 1 by 5 plus minus 7 into minus 1 by 5 is minus M by 5 plus 7 by 5. So, this will become 7 minus M by 5 with a minus sign. So, I will have a minus sign here. So, I do not have a minus sign just erase it. So, I get minus M by 5 and 0 minus M by 5 is plus M by 5, which comes here and here I have plus 7 by 5. So, 0 minus 7 by 5 is minus 7 by 5. So, m minus 7 by 5 is correct.

Fifth 1, minus M into 1 is minus M plus 0. So, I will have a 0 there, I will have a 0 here, because it is a basic variable, for the last one, plus M by 5 minus 7 by 5. So, minus M minus M by 5 plus 7 by 5 will become minus 6 M plus 7 by 5. The value of the objective function is again written as dash, because there is still an artificial variable in the solution. So, it becomes dash.

Now, again we want to check whether we have to stop here, whether the optimum solution is reached or whether we need to proceed further. So, the most positive C j minus Z j will enter. So, this is negative, this is negative, 3 M minus 11 by 5, M minus 7 by 5, 3 M is bigger than M, therefore variable X 2 will try to come into the solution. Now, when variable X 2 tries to come into the solution, it has to replace either a 1 or X 1.

To do that we find out the limiting values, 2 divided by 3 by 5 is 2 into 5 by 3, which is 10 by 3, 2 divided by 2 by 5 is 2 into 5 by 2, which is 10 by 2, which is 5, the smaller value is 10 by 3, so that is the limiting value. So, variable X 2 will come in and variable a 1 will go out. Now, I have shown the entire thing both the tables I have shown, variable X 2 will come into the solution and variable a 1 will go out of the solution. So, this number shown in a different color is the pivot element.

So, we now write the new basic variables, which happened to be X 2 and X 1, X 2 has come in and has replaced a 1. So, we write X 2 and X 1, the objective function values are minus 5 and minus 7 from this part of the table. Now, we have to write this row first, the pivot row has to be replaced. So, divide every element of the pivot row by the pivot element. So, the first row is written by dividing every element by 3 by 5.

So, 0 divided by 3 by 5 is 0, 3 by 5 divided by 3 by 5 is 1, minus 1 divided by 3 by 5 is minus 5 by 3, 1 by 5 divided by 3 by 5 is 1 by 3, 1 divided by 3 by 5 is 5 by 3 and minus 1 by 5 divided by 3 by 5 is minus 1 by 3, 2 divided by 3 by 5 is 10 by 3. So, the first row is written. Now, we have to write the second row. Now, to do that, we have this variable X 2 coming here. So, this should have a 0 here, it should have a 1 0, X 1 should have a 0 1, X 2 is the 1 that has come in now. So, I should get a 0 here.

So, to get a 0 here I look at the corresponding position, which is 2 by 5. So, this minus 2 by 5 times 1 is 0. So, 2 by 5 minus 2 by 5 times 1 is 0. So, we use that 1 minus 2 by 5 times 0 is 1, 2 by 5 minus 2 by 5 times 1 is 0, 0 minus 2 by 5 into minus 5 by 3 is plus 2 by 3, minus 1 by 5, minus 2 by 5 into 1 by 3 is minus 1 by 5 minus 2 by 15. So, minus 1 by 5 is minus 3 by 15, minus 2 by 15 is minus 5 by 15, which will become minus 1 by 3, 0 minus 2 by 5 into 5 by 3 is minus 2 by 3. And 1 by 5 minus 2 by 5 into minus 1 by 3 is 1 by 5 plus 2 by 15, which will become plus 1 by 3, 2 minus 2 by 5 into 10 by 3 is 2 minus 4 by 3. So, 2 minus 4 by 3 is 2 by 3, which is shown here.

Now, we have to check whether this solution is optimum or we have reached the optimum here or do we need to go further. To do that, we have to find the C j minus Z j values. So, we do C j minus Z j. So, minus 5 into 0 plus minus 7 into 1 is minus 7, minus 7, minus, minus 7 is 0, Z 2 is minus 5 plus 0 is minus 5, C 2 minus Z 2 is minus 5, minus, minus 5, which is 0. So, the basic variables will have 0 value.

Here, it is plus 25 by 3 minus 14 by 3 which is plus 11 by 3, 0 minus 11 by 3 is minus 11 by 3, here it is minus 5 by 3 plus 7 by 3, which is plus 2 by 3. So, 0 minus 2 by 3 is minus 2 by 3. Now, here it is minus 25 by 3 plus 14 by 3, which is minus 11 by 3, so minus M plus 11 by 3. So, it is minus M plus 11 by 3, which is minus 3 M by 3 plus 11 by 3. The last one is plus 5 by 3 minus 7 by 3 is minus 2 by 3. So, it becomes minus M minus 2 by 3 is minus M plus 2 by 3. So, this will become minus 3 M plus 2 by 3.

The value of the objective function now we calculate minus 5 into 10 by 3 minus 50 by 3 minus 14 by 3 is minus 64 by 3. Now, we need to check whether, there is any positive C j minus Z j. Now, we have 0, 0, negative, negative minus 3 M plus 11 by 3 is a negative quantity, minus 3 M plus 2 by 3 is also a negative quantity. So, all of them are negative, no variable is entering, therefore the algorithm stops.

By giving us a solution X 1 equal to 2 by 3, X 2 equal to 10 by 3, objective function equal to minus 64 by 3. But, we have to remember that we have converted a minimization problem to a maximization problem by multiplying with a minus 1. Therefore, we have to multiply back with a minus 1 for the minimization and say that, the objective function value for the minimization is plus 64 by 3, while for the corresponding maximization, it was minus 64 by 3.

So, when we solve a minimization problem as a maximization problem. We will get a negative value and finally, we have to make it positive. So, the optimum solution is X 1 equal to 2 by 3, X 2 equal to 10 by 3 and objective function is 64 by 3 for the minimization problem. So, this is how is the simplex table works for the minimization problem, when we have greater than or equal to type constraints, we will have negative slack variables, which cannot become starting variables for the solution.

So, we add artificial variables and when we add artificial variables, we will have this involving the minus M for maximization and plus M for minimization. So, this method is also called big M method, because we are using the big M in solving these problems. So, whenever we have greater than or equal to constraints, we will be adding the artificial variables.

And most minimization problems will have greater than or equal to constraints and therefore, we use this big M method to solve minimization problems using the simplex algorithm. Other aspects of the simplex table, we will now see in subsequent classes.