Introduction to Operations Research
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Module - 03
Simplex Algorithm
Lecture - 01
Algebraic form of Simplex Algorithm

In this module, we will be introducing the Simplex Algorithm. The simplex algorithm is the most important algorithm to solve linear programming problems.

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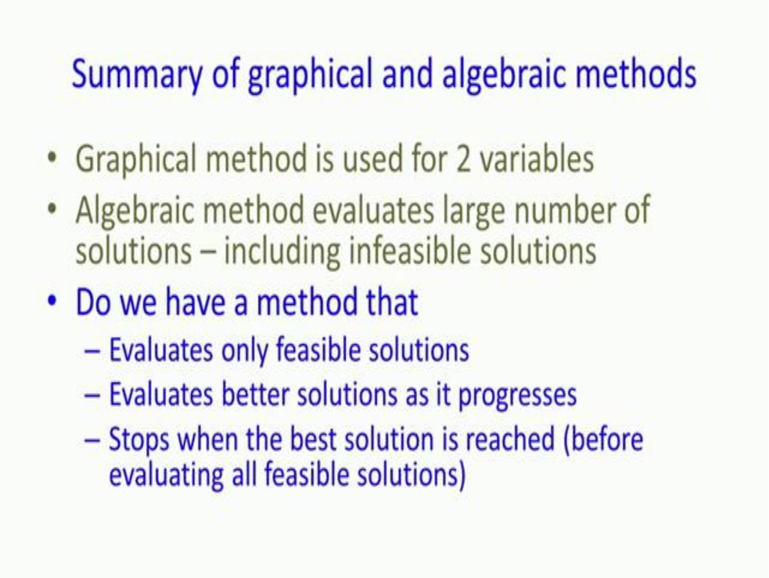

Let us look at the summary of the graphical and the algebraic methods with which we ended the previous class. We have seen the graphical and algebraic methods in the previous module and we understand that the graphical method is useful, when we have two variables. The algebraic method can be used, when we have more than two variables or large number of variables, but it evaluates a large number of solutions.

There are times it evaluates infeasible solutions also and we do not have a way to understand a priory that this solution is going to be infeasible. So, we have to evaluate it and then, understand that it is infeasible. So, in some ways, there are three aspects of the algebraic method which if we can correct using a better algorithm, we can solve linear programming faster and better.

Now, the algebraic method evaluates infeasible solutions. So, is there a way to eliminate infeasible solutions? Secondly, is there a way to evaluate better solutions as we move a long and the third is, is there a way we understand that we have reached the best solution and we do not have to evaluate anymore? So, these three questions remain at the end of the algebraic method.

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Algebraic Method

Maximize $10X_1 + 9X_2 + 0X_3 + 0X_4$
Subject to
$3X_1 + 3X_2 + X_3 = 21$
$4X_1 + 3X_2 + X_4 = 24$
$X_1, X_2, X_3, X_4 \geq 0$

No.	Variables solved (Basic variables)	Variables fixed to zero (non basic variables)	Solution	Objective function value	Comments
1.	X_3 and X_4	X_1 and X_2	$X_3 = 21$, $X_4 = 24$	$Z = 0$	Basic feasible
2.	X_1 and X_3	X_2 and X_4	$X_1 = 6$, $X_3 = 3$	$Z = 60$	Basic feasible
3.	X_1 and X_4	X_2 and X_3	$X_1 = 7$, $X_4 = -4$		infeasible
4.	X_2 and X_3	X_1 and X_4	$X_2 = 8$, $X_3 = -3$		infeasible
5.	X_2 and X_4	X_1 and X_3	$X_2 = 7$ $X_4 = 3$	$Z = 63$	Basic feasible
6.	X_1 and X_2	X_3 and X_4	$X_1 = 3$, $X_2 = 4$	**$Z = 66$**	Basic feasible - **optimum**

Let us go back to the same example in the algebraic method, we evaluated six solutions and we did not have a way to know that our solutions 3 and 4 are going to be infeasible, before we evaluated them. Now, is there a way not to evaluate them at all and evaluate only the remaining four basic feasible corner points is the first question. Second observation is, if we look at the four corner points, we seem to have evaluated them in the order, where the solution gets better and better.

Now, let us assume that we had evaluated number 5 and we have a solution with 63 and let us assume that we have not evaluated the second solution. Now, is there a way not to evaluate the second solution? Because, even if we evaluated the second solution, we are going to have 60 as the objective function value and since, we are maximizing the objective function, where will not be a way by which solution number 2 can be optimum, if we know that, there is a solution with 63.

So, it so happened here that we evaluated better and better solutions, but can we make it a distinct part of the algorithm. We have not made it a part of the algorithm, can we make it a part of the algorithm. Now, instead of evaluating the sixth solution last, if we had evaluated as a fourth or fifth solution, is there a way to understand at that point that I have reached the optimum and I do not have to evaluate anymore. So, these three issues we try to address in an algebraic way and later, we extend that to what is called the simplex algorithm.

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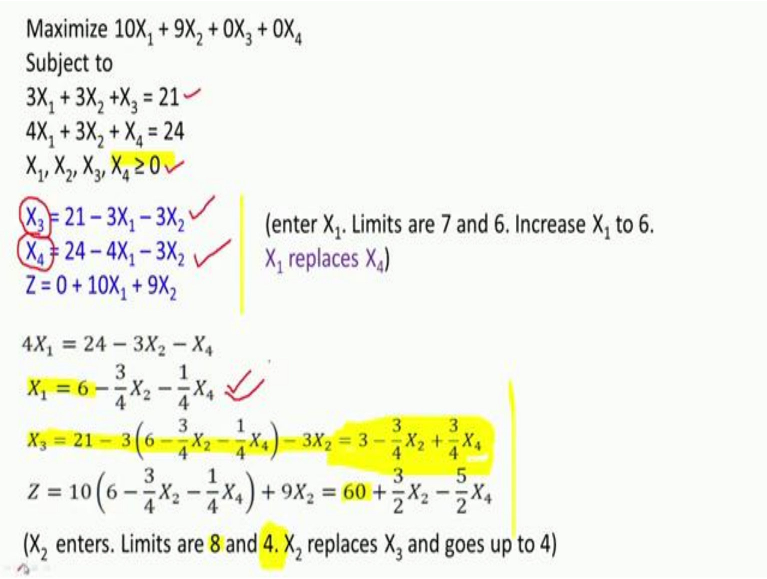

So, let us go back to the same example, where we have added the slack variables. There are two slack variables X 3 and X 4, they do not contribute to the objective function, they also are greater than or equal to 0. Now, let us assume that we want to start the algebraic method in a certain way, because there are two equations and four variables. We can only solve for two variables at a time and those two variables, we are going to solve are called basic variables. It also means that the remaining two variables have to be fixed and they are fixed to 0, they are the non-basic variables.

So, by simply looking at these two equations a very easy way to start is, by treating X 3 and X 4 as basic variables, which means X 1 and X 2 are non basic variables and they have value 0. So, we start with X 3 and X 4 as basic variables and we write the equations in such a way that they are written as basic variables in terms of the non basic variables. The first equation is 3 X 1 plus 3 X 2 equal to 21, now from this equation, we are writing X 3 is equal to 21 minus 3 X 1 minus 3 X 2.

Now, X 3 is your basic variable, X 3 is a basic variable. Now, the second equation is 4 x 1 plus 3 X 2 plus X 4 equal to 24, this is written as X 4 is equal to 24 minus 4 X 1 minus 3 X 2. So, X 4 is your basic variable. The objective function is 10 X 1 plus 9 X 2 or 0 plus 10 X 1 plus 9 X 2, the 0 comes because X 3 and X 4 do not contribute to the objective function.

Now, from these two equations, where we have written the basic variable in terms of the non basic variable, the non basic variables take value 0. Therefore, the solution is X 3 equal to 21 X 4 equal to 24, which is one of the basic feasible solutions that we got through the algebraic method. So, we have the starting solution X 3 equal to 21, X 4 equal to 24, X 1, X 2 equal to 0 is basic feasible, because it satisfies the non negativity restriction, it also satisfies the two given equations.

So, now, we have written the basic variables in terms of the non basic variables. Now, X 3 equal to 21 X 4 equal to 24 would give a contribution of 0 to the objective function plus 10 X 1 plus 9 X 2. We still write 10 X 1 plus 9 X 2, X 1 and X 2 are right now at 0. Now, we have a solution with X 1 equal to 0, X 2 equal 0 and Z equal to 0. Now, the objective function has a value 0 and we wish to maximize the objective function.

Now, this Z which is the objective function can be increased, if we increase X 1 or we increase X 2, both X 1 and X 2 are at 0. So, if we increase X 1, the rate of increase is 10 and if we increase X 2, the rate of increase is 9. So, we start with increasing X 1, because it has a larger rate of increasing the objective function. So, we increase X 1. So, when we increase X 1, we call it by saying we enter X 1.

Now, as we increase X 1, let us say X 1 is at 0, now if we make X 1 equal to 1, then X 3 will become 18 from 21, X 4 will become 20 from 24. If we increase X 1 to 2, then X 3 will become 15 and X 4 will become 16. As X 1 increases, X 3 and X 4 start reducing, because of the negative sign and as we keep increasing X 1, one of them will quickly first become 0. So, this for X 3 to become 0, we have to increase X 1 to 7, for X 4 to become 0, we have to increase X 1 to 6.

So, as we keep increasing X 1, when X 1 reaches 6, X 4 become 0 first and if we increase X 1 further, X 4 will become negative, which we do not want. Therefore, we say that X 1 can be increased up to 6. So, X 1 is increased up to 6 and X 1 is now going to become basic, because it is not having a value 0, X 4 which was 24, now becomes 0 and will become non basic. Therefore, we write the equation with X 1 as a basic variable and X 4 as a non basic variable.

So, this equation this equation is now rewritten as this equation, which gave us X 1 equal to 6 is now rewritten as 4 X 1 is equal to 24 minus 3 X 2 minus X 4 and this is simplified and again written as X 1 is equal to 24 by 4, 6 minus 3 by 4 X 2 minus 1 by 4 X 4. So, now, X 1 becomes basic with value equal to 6. Now, we have to find out the value of X 3 and we do that by substituting for X 1 in the first equation or in the equation for X 3.

So, we now use this value of X 1 that we have taken and we substitute it in this X 3 equation to get X 3 is equal to 21 minus 3 times X 1, X 1 has been recently written minus 3 X 2. So, when we do this simplification, we will get X 3 is equal to 3 minus 3 by 4 X 2 plus 3 by 4 X 4, 21 minus 18 will give that three we have plus 9 by 4 X 2 minus 3 X 2. So, minus 3 X is minus 12 by 4. So, we have minus 3 by 4 X 2 and then, we have plus 3 by 4 X 4 from this equation.

And then, we now write; now we now write the objective function as it was originally 0 plus 10 X 1 plus 9 X 2. So, the objective function becomes 0 plus 10 times X 1, X 1 is 6 minus 3 by 4 X 2 minus 1 by 4 X 4. So, 10 times 6 minus 3 by 4 X 2 minus 1 by 4 X 4 plus 9 X 2. So, 10 into 60 minus 30 by 4 plus 9 minus 30 by 4 plus 36 by 4 is plus 6 by 4, which is 3 by 2 X 2 and 10 minus 10 by 4 is minus 5 by 2 X 4. So, from this solution, from this solution, from this solution, the first solution, we have now moved to another solution that we have written here.

Now, this second solution has X 1 equal to 6 and X 3 equal to 3 and the objective function value is equal to 60, 10 into 6 is 60. So, we have moved to another solution right now. Now, again we wish to maximize the objective function, the objective function is now 60 plus 3 by 2 X 2 minus 5 by 2 X 4, X 2 and X 4 are non basic. So, both of them are at 0 and therefore, the value of the objective function is 60.

Now, Z can be increased further, if we increase X 2, because X 2 now has a positive rate of increasing Z, the coefficient is positive. So, it has a positive rate of increasing Z. Now, by increasing X 4, we cannot increase Z, we actually end up decreasing Z, because of the negative coefficient. Because, it has a negative coefficient, the only way by which we can increase Z through X 4 is by decreasing X 4, because X 4 is at 0.

But, if we decrease X 4, X 4 will become negative, which will violate the non negativity restriction, which is here. Therefore, we do not consider X 4 as a way to increase Z. We look at only X 2 as a way to increase Z and since, X 2 has positive coefficient, it is at 0. So, if we increase Z, we can further increase Z or increase the objective function. Now, we see the extent to which we can increase this X 2.

Now, as X 2 increases from this equation, you realize that X 1 is going to reduce, as X 2 for example, if X 2 becomes 1, then X 1 will become 6 minus 3 by 4. So, it reduces. So, this equation would allow X 2 to increase up to 6 into 4 divided by 3, it would allow X 2 to go up to 8, it would allow X 2 to go up to this 8, again repeating 6 and 3 by 4. So, the best value that X 2 can take till X 1 become 0 is 6 into 4 by 3, which is 8, at X 2 equal to 8, 3 by 4 X 2 will become 6 and X 1 will become 0.

This equation we have to look at this. So, this become 3 as X 2 increases as X 2 becomes one this will become 3 minus 3 by 4 and. So, on this would allow X 2 to go up to 12 by 3 4. So, we have X 2 going up to 4. So, now, as X 2 becomes 4, this equation will make X 3 come to 0 first and therefore, the limiting value is a smaller of the two values. So, we will now say that X 2 will replace X 3 from this equation, because this equation allows X 2 to go up to 4, which is here, which is here.

So, this is the equation that we will be using particularly this equation, which we will be using. Now, X 2 will come to the left hand side of this equation, X 3 will become 0 and go to the right hand side of this equation.

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In previous iteration $X_3 = 3 - \frac{3}{4}X_2 + \frac{3}{4}X_4$

$$\frac{3}{4}X_2 = 3 - X_3 + \frac{3}{4}X_4$$

$$X_2 = 4 - \frac{4}{3}X_3 + X_4$$

$$X_1 = 6 - \frac{3}{4}\left(4 - \frac{4}{3}X_3 + X_4\right) - \frac{1}{4}X_4 = 3 + X_3 - X_4$$

$$Z = 60 + \frac{3}{2}X_2 - \frac{5}{2}X_4 = 60 + \frac{3}{2}\left(4 - \frac{4}{3}X_3 + X_4\right) - \frac{5}{2}X_4$$
$$= 66 - 2X_3 - X_4$$

So, from the previous solution X 3 was 3 minus 3 by 4 X 2 plus 3 by 4 X 4, X 2 comes to the left hand side and X 3 comes to the right hand side. Now, this is written as 3 by 4 X 2 is equal to 3 minus X 3 plus 3 by 4 X 4 and X 2 is written as 4 minus 4 by 3 X 3 plus X 4 from this you multiply everything by 4 by 3, you get this expression. Now, go back to the previous one and rewrite X 1 in terms of X 2. So, X 1 is 6 minus 3 by 4 X 2 minus 1 by 4 X 4. So, we substitute for X 2 from this equation, we substitute for X 2 into X 1 and we get three plus X 3 minus X 4.

Now, once again we go back and write the objective function now, the objective function was 60 plus 3 by 2 X 2 minus 5 by 2 X 4. So, we go back and substitute for X 2 from this equation into the objective function and rewrite it and simplify it to get 66 minus 2 X 3 minus X 4. So, we now have another solution, which is X 2 equal to 4, X 1 equal to 3 and objective function equal to 66, X 3 and X 4 are non basic, they are at 0.

Now, we wish to maximize Z further and we realize that Z can be increased beyond 66, only if we decrease X 3 and X 4, because both have negative coefficients. And if we decrease them, they will become negative, because they are at 0. So, we do not try to decrease X 3 and X 4. So, we do not have a variable which can now help us increase the value of the objective function.

We now stop the algorithm by saying that we have reached the best solution and the best solution is X 1 equal to 3, X 2 equal to 4 with Z equal to 66. So, now we have shown or developed or used an algorithm which essentially does what we wanted to do. Now, this algorithm did not evaluate any infeasible solution, it evaluated three solutions. But, it did not evaluate any infeasible solution; it evaluated only basic feasible solutions. It evaluated 3 out of the 4 basic feasible solutions and it also gave solutions with better values.

The objective function value was 0, 60 and 66, when it reached 66, it knew that it had reached the optimum; it did not evaluate the fourth one. So, this way, this algorithm is able to meet our requirement of refining the algebraic method and making it better. This is the bases of what is called the simplex algorithm and we will see the simplex algorithm in the form of a table in the next class.