Introduction to Operations Research
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Module – 02
Graphical and Algebraic methods
Lecture - 05
Comparing Graphical and Algebraic methods

In this class, we summarize the Graphical and Algebraic methods and we relate the two methods, which are popular to solve linear programming problems.

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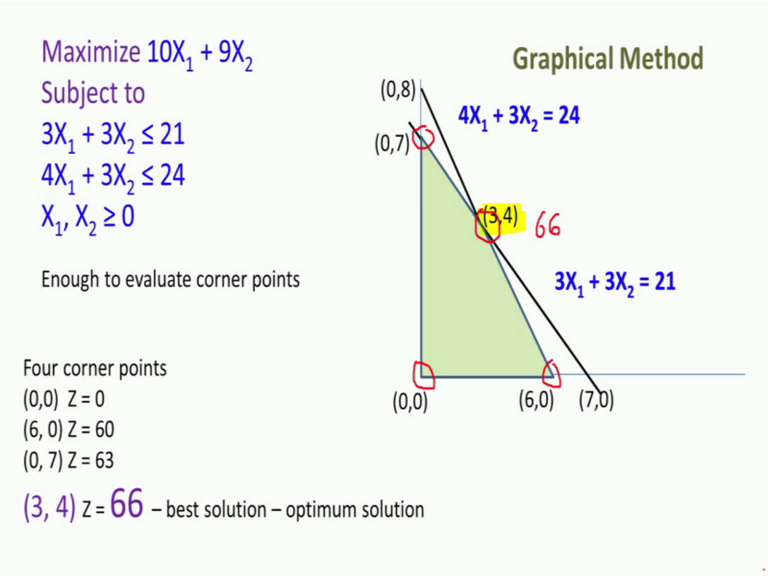

We started with a maximization problem and we solved it using the graphical method and then, in another class, we took a minimization problem and solved it using the graphical method. In the last two classes, we have solved the same two examples, one for maximization and one for minimization using the algebraic method. Now, let us go back to the graphical method and look at what we did and then, we go back to the algebraic method and try to see, whether there is a relationship between what we did in the graphical method and what we did in the algebraic method.

Now, in the graphical method, we found out the feasible region and then, we explained that for every point inside the feasible region, there is always a boundary point or a corner point, which will give a better solution, a better value of the objective function.

And then, we restricted ourselves to evaluating only the corner point solutions and we found the optimum solution from the best corner point solution.

In this example, we had these four corners points 0 comma 0, 6 comma 0, 0 comma 7 and 3 comma 4, which are our four corner points and we said that the optimum solution is this corner point 3 comma 4, which give us a value of 66. Now, let us go back to the corresponding algebraic method and let us see, what we did.

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Algebraic Method

Maximize $10X_1 + 9X_2 + 0X_3 + 0X_4$
Subject to
$3X_1 + 3X_2 + X_3 = 21$
$4X_1 + 3X_2 + X_4 = 24$
$X_1, X_2, X_3, X_4 \geq 0$

No.	Variables solved (Basic variables)	Variables fixed to zero (non basic variables)	Solution	Objective function value	Comments
1.	X_3 and X_4	X_1 and X_2	$X_3 = 21$, $X_4 = 24$	Z = 0	Basic feasible
2.	X_1 and X_3	X_2 and X_4	$X_1 = 6$, $X_3 = 3$	Z = 60	Basic feasible
3.	X_1 and X_4	X_2 and X_3	$X_1 = 7$, $X_4 = -4$		infeasible
4.	X_2 and X_3	X_1 and X_4	$X_2 = 8$, $X_3 = -3$		infeasible
5.	X_2 and X_4	X_1 and X_3	$X_2 = 7$ $X_4 = 3$	Z = 63	Basic feasible
6.	X_1 and X_2	X_3 and X_4	$X_1 = 3$, $X_2 = 4$	**Z = 66**	Basic feasible - **optimum**

Now, this is the table corresponding to the six solutions for the same linear programming problem, which maximize 10 X 1 plus 9 X 2 plus 0 X 3 plus 0 X 4 subject to these constraints. Now, as I mentioned with the inclusion of this slack variable, there are four variables, out of which we can only solve for two at a time and therefore, we had 4 C 2 or six ways of solving these solutions.

Now, these six solutions are given here for the six problems instances and let us look at these solutions, only from the point of view of X 1 and X 2 and relate it to X 1 and X 2, because the graphical method, we used only X 1 and X 2. So, the first solution that we have here is X 1 equal to 0 and X 2 equal to 0. So, the first solution that we have here is X 1 equal to 0 and X 2 equal to 0, because X 3 and X 4 have some values.

The second solution that we have here is X 1 equal to 6, X 2 equal 0, which is 6 comma 0. The third solution that we have here is X 1 equal to 7 and X 2 equal to 0, which is a 7 comma 0. The fourth solution that we have is X 1 equal to 0, X 2 equal to 8. The fifth solution that we have is X 1 equal to 0, X 2 equal to 7 and the sixth solution that we have is X 1 equal to 3 and X 2 equal to 4. So, these are the six solutions that we have.

Now, out of these six solutions, we also said that situations 3 and 4 are infeasible. Now, we also have to remember that all the six solutions are basic solutions, which means that the non basic variables, which are here which are X 1, X 2, X 2, X 4, etcetera are all fixed to 0. Even though, they could have been fixed to any other value, these six solutions have been obtained by fixing the non basic variables to 0. So, now, let us go back to the graphical method and try to see, where these six solutions lie in the graph that we actually have. So, let us do that.

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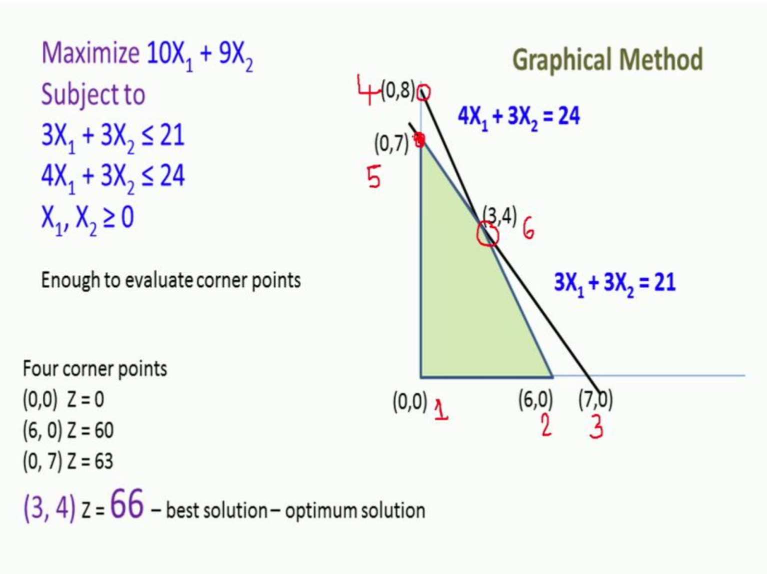

The first solution is X 1 equal to 0 and X 2 equal to 0. So, we denote this first solution, which is X 1 equal to 0 and X 2 equal to 0 as the first solution that we have. The second solution out of the 6 is X 1 equal to 6 and X 2 equal to 0. So, X 1 equal to 6 X 2 equal to 0 is our second solution. The third solution is X 1 equal to 7 and X 2 equal to 0, which happened to be infeasible in the algebraic method.

So, X 1 equal to 7, X 2 equal to 0 is the third solution, which is here and you can observe from the graph that this solution 7 comma 0 is not within the feasible region and therefore, it is infeasible. The fourth solution is X 1 equal to 0 and X 2 equal to 8, which was also infeasible. Now, we go back and mark it in the graph and we realize that this point, this is the fourth solution, so this is marked as 4.

The fifth solution was X 1 equal to 0 and X 2 equal to 7 and we mark this as the solution, which is here, this is the solution X 1 equal to 0 and X 2 equal to 7 is marked as the fifth solution. And the sixth solution is X 1 equal to 3 and X 2 equal to 4, which is marked as the sixth solution marked. Now, we observe that the six basic solutions that we obtained in the algebraic method are actually six corner points in the graph.

And since, two of these corner points, the points where these lines intersect two of them are infeasible, ((Refer Time: 08:46)) these two are now shown as infeasible solutions. Now, when we look at the algebraic method and when we look at these six solutions, we have to leave out the infeasible ones. In this example, there are two infeasible solutions, we leave out the infeasible solutions, we consider only the basic feasible solutions and we considered only the basic feasible solutions and found out the best basic feasible solution out of the 4.

Now, this is exactly the same as, we now realize that the four feasible or four basic feasible solutions are actually the four corner points which are points 1, 2, 5 and 6. Therefore, what we did in the algebraic method is the same as what we did in the graphical method. Now, in the graphical method, we were restricted to two variables and we were able to clearly look at the graph and say that these two solutions are infeasible. By looking at the graph, we can do that and therefore, we did not consider them for evaluating the objective function.

We looked only at the corner points and we defined those corner points, which are basic feasible solutions. So, the first relationship is that, the algebraic method evaluates the intersection points that the graphical method has. Now, out of these six intersection points that we have, two of them are infeasible and four are basic feasible. The algebraic method also evaluates the same six points and declares two of them that are infeasible and the rest of them are basic feasible.

The graphical method looks at the four corner points, which happened to be the four basic feasible solutions. So, every corner point in the graphical solution corresponds to a basic feasible solution in the algebraic method. So, when we say that the graphical method has four corner points and the algebraic method has four basic feasible solutions, it is enough, they are one and the same. Let us now go back to another aspect in the algebraic method.

Now, let us take this particular solution, where we are solving for X 1 and X 2. Now, we are fixing X 3 and X 4 to 0 and we are solving. Now, when we are fixing X 3 and X 4 to 0, then the equations become 3 X 1 plus 3 X 2 equal to 21 and 4 X 1 plus 3 X 2 equal to 24. So, X 1 and X 2 is the solution of 3 X 1 plus 3 X 2 equal to 21, 4 X 1 plus 3 X 2 equal to 24.

Now, that is exactly the solution here, which is between 3 X 1 plus 3 X 2 equal to 21; that gives us the 3 comma 4; that is the solution between 3 X 1 plus 3 X 2 equal to 21 and 4 X 1 plus 3 X 2 equal to 24. So, fixing the non basic variable to 0, gives us the point of intersection of the lines, which is the corner point. If we had fixed X 3 and X 4 to 1, say just for the sake of illustration and not fixed it to 0.

Then, we would have solved for 3 X 1 plus 3 X 2 equal to 20 and 4 X 1 plus 3 X 2 equal to 23 and we would have got a different solution and we would have got a point inside the feasible region. So, fixing the non basic variables to any positive value is equivalent of evaluating a point inside the feasible region while, fixing the non-basic variables to 0 is equivalent of evaluating the corner point. And from the graphical method, we have already learnt that points inside the feasible region need not be evaluated.

Now, this implies that when we do the algebraic method, it is enough to fix the non basic variables only to 0 and not fix them to any other value. Fixing them to any other value would end up evaluating points, which are inside the feasible region or fixing them to negative values would mean evaluating infeasible points. Therefore, the understanding and the equivalent between the graphical and the algebraic method, tells us that, since corner point solutions are optimal. It is enough to fix the non basic variables to 0 in the algebraic method and not to fix them at any other positive value.

So, now, we understand by considering the same example, we understand that the graphical method and the algebraic method, essentially try to do the same thing. But, the graphical method is simpler, ((Refer Time: 14:19)) because we are used to drawing graphs and noticing the corner points, but the graphical method has a restriction; that it can be used only up to two variables.

Now, when we have a large number of variables, we can use the algebraic method to solve. We have illustrated the algebraic method by considering a two variable problem, but we can also illustrate the algebraic method by considering more variables. If the original problem itself had three variables and two constraints, then when we add the slack variables, we would have got five variables and then, two constraints. So, we have to solve for two at a time. So, we will have 5 C 3 possible solutions, we will have more number of solutions as the number of variables and constraints increase.

Now, out of these solutions, we have to evaluate all of them and we have to leave out the infeasible ones and look only at the basic feasible ones. To begin with, we have to understand that we do not have a way to identify that something is going to result in an infeasible solution. So, we have to evaluate all the solutions and if we have n variables and m equations, we end up creating n C m solutions.

In this example n is equal to 4, there are four variables, m is equal to 2, there are two constraints, 4 C 2 is equal to 6. Now, if we go back to the minimization problem. Now, when we solved the minimization problem, we also had six solutions in the algebraic method and three of these solutions were infeasible, while three of these solutions were basic feasible.

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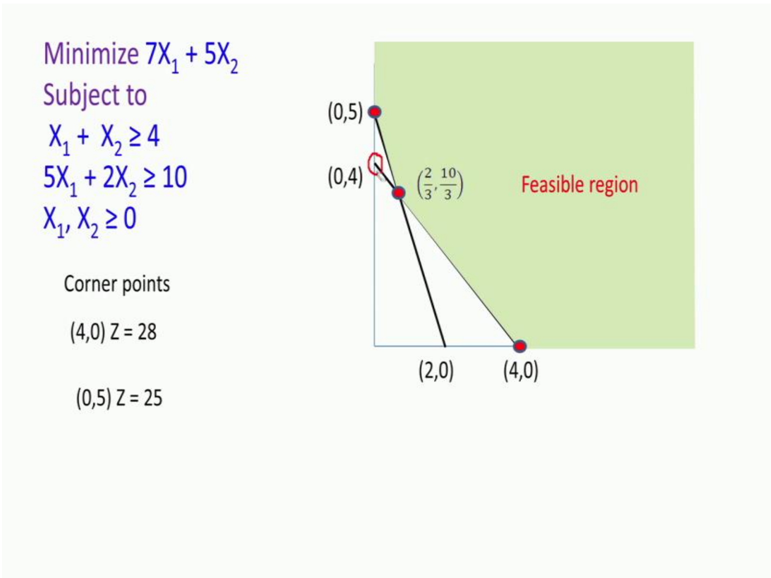

If we go back and look at the graph corresponding to the minimization problem, now we realize that, these are the three solutions that are feasible 4 comma 0, 0 comma 5 and 2 by 3 comma 10 by 3. These were the solutions that are feasible. The other three solutions that the algebraic method evaluated, where here 0 comma 0, 2 comma 0 and 0 comma 4, it evaluated and these three became infeasible. So, out of these six solutions, we looked at only three solutions in this and those three solutions are the three corner points in the feasible region.

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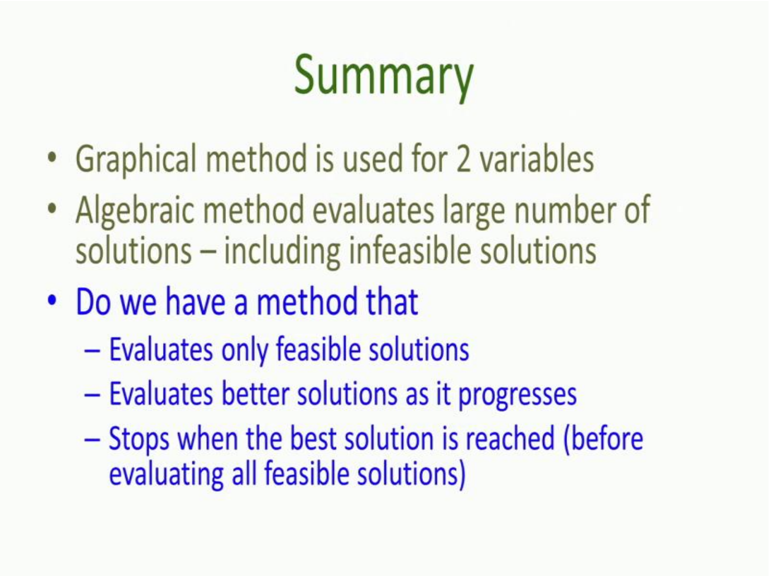

Now, in summary, we observe that the graphical method can be used only for a two variable problem. The algebraic method can evaluate a large number of solutions and sometimes evaluates infeasible solutions. We observed that for the maximization problem, 2 out of the 6 were infeasible, for the minimization problem, 3 out of the 6 were infeasible.

So, that leads to the next question, do we have a method that evaluates only feasible solutions, does not evaluate infeasible solutions. Do we have a method that progressively evaluates better solutions and do we have a method that can stop and tell us that the best solution has been obtained even before evaluating all possible basic solutions? So, we end this discussion on graphical and algebraic with this question that do we have a method that can do these three things. The answer to this question, we will see in the next class.