Investment Management
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Lecture - 21
Two - Fund Theorem

Hi there. We are discussing the course Investment Management and in this module, we are going to talk about construction of funds particularly in the context of mutual funds. And, before to start with in this particular session we are going to discuss about Two -Fund Theorem.

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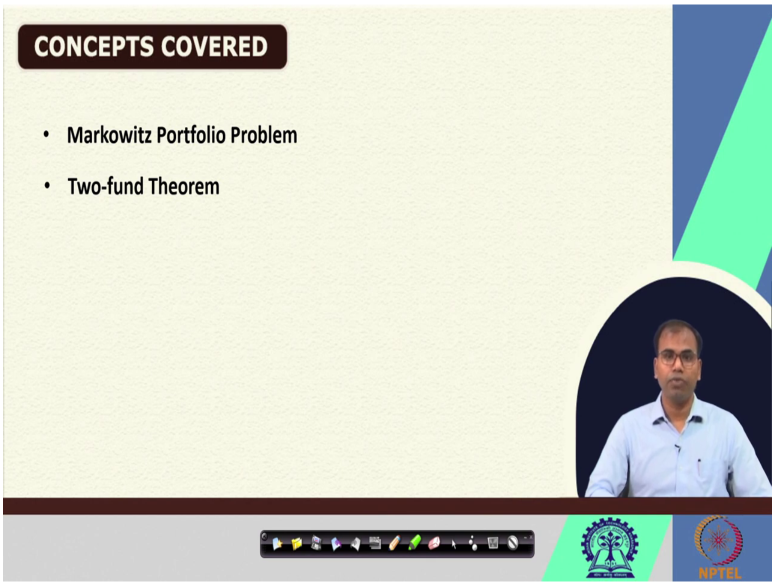

What essentially, we are going to do in this particular session is we will start with the basic assumption of Markowitz portfolio problem, which is essentially an optimization problem for finding out the weights of any individual asset to be included as part of the portfolio. And, then from there we will try to develop our understanding of a theory called two - fund theorem.

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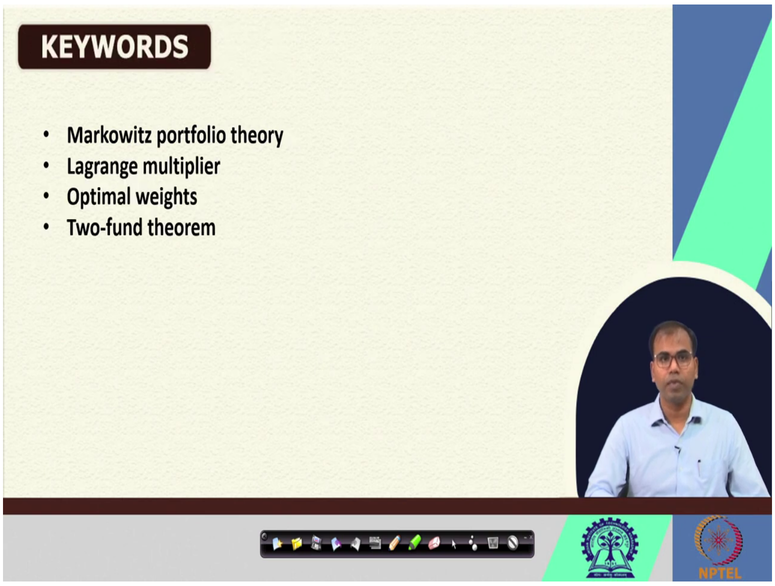

So, essentially, we will begin with Markowitz portfolio theory. We will use the Markowitz assumption and to find the optimal weights and then we will conclude with our discussion on two fund theorem.

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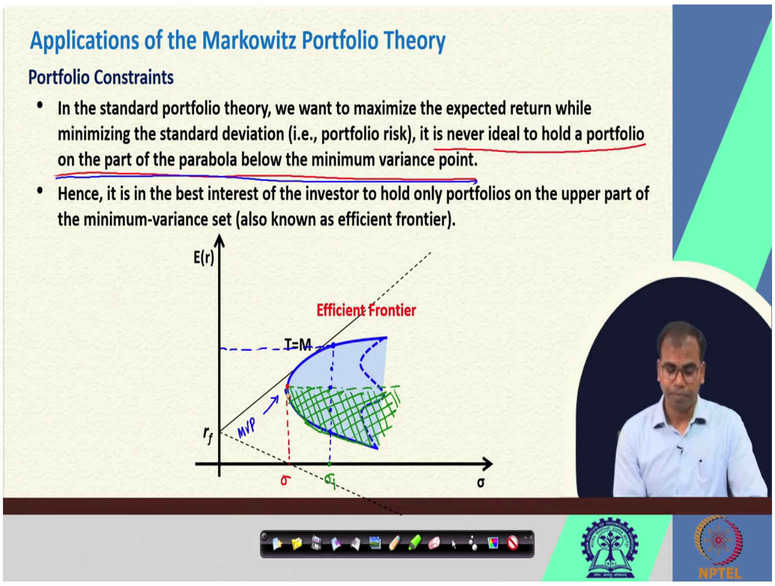

As we know so far in the standard portfolio theory, an investor wants to maximize the expected return while reducing or minimizing the standard deviation which is an indicator of portfolio risk. And, it is never ideal to hold a portfolio on the part of the parabola below the minimum variance point. If you try to look at this graph or this portfolio visualization, where we know that an investor would want to hold a portfolio where she can get the highest possible return for a given level of risk.

So, if you try to understand this statement using this two-dimensional graph, where we know that for an investor it is never ideal to hold a portfolio on the part of the parabola below the minimum variance point. So, first let us understand this minimum variance point.

We know that if this is the parabola or this is the combination of asset, this is the point where the risk is minimum. So, this particular point on this parabola is basically the minimum variance point, because an investor does not have any portfolio available beyond this before this point to offer any risk that is anything less.

Now, an investor is not interested in buying any invest portfolio or any investment with the lower range of the parabola, which means this part of the parabola although it offers same level of risk, but it offers lower return. And, that is why no investor would like to invest her money in this part of the this part of the entire efficient region.

So, basically this is a no go zone for an investor, because we have already discussed for an investor if this is the desired level of risk sigma i, the investor has a choice to invest her money in a portfolio, that lies here. It can lie here as well, it can lie here as well, it can lie here.

But it is always in the interest of the investor to have a portfolio here because for the same level of risk, the investor is getting the highest level of return at this point of time. And, that is what this statement suggest that investor would not like to hold a portfolio on the part of the parabola below the minimum variance point.

And, that is why it is always in the best interest of the investor to hold only portfolios on the upper part of the minimum variance set which we know as efficient frontier. Now, this we have already discussed how we get this efficient frontier and how the choice of holding a portfolio with unique combination of risk and return made by the investor. Now, let us try to understand it mathematically.

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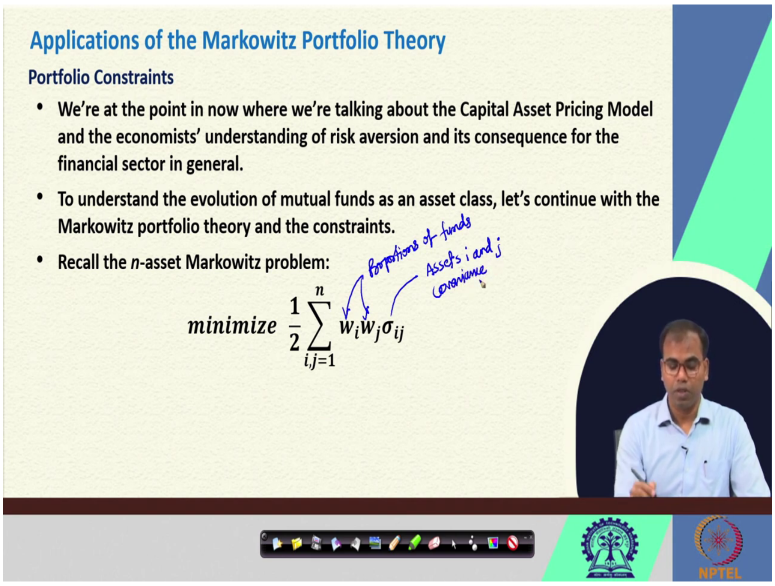

So, as of now we are at the point where we know about capital asset pricing model, we have been discussing about capital asset pricing model for quite some time. And, we also know some sort of understanding of risk aversion and the consequences of risk aversion for the financial sector in general.

We know that majority of the investors would act as risk averse; they do not want to hold any marginally risky portfolio unless they are paid or they are supplied with additional return. If they expect more return, then only they would like to hold more risky portfolios. So, to understand the concept of mutual fund as an asset class, we need to begin with understanding the Markowitz portfolio theory and the constraint that is used for developing this Markowitz portfolio problem.

Now, we know that the original Markowitz portfolio problem, if we could recall this n asset Markowitz problem of portfolio optimization, it suggest or it is essentially about minimizing the combined risk of individual assets such that the return is maximum. Now, this is the objective function under Markowitz portfolio problem, where we have this covariance variance of asset i and j.

And, here we have respective weights or proportion of funds proportion of funds that are invested in these two assets. So, with this objective function we know that like any other objective function this objective function also operates with certain constraint.

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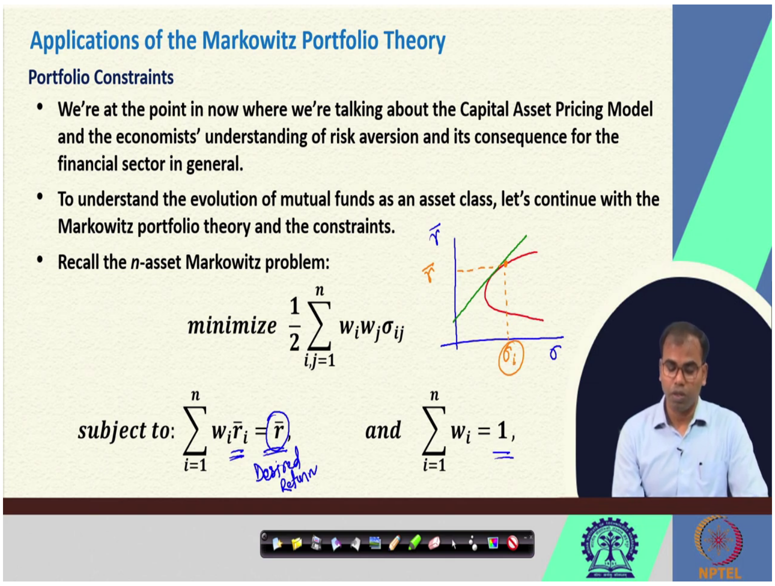

And, the first constraint is the rate of return that is desired by the investor needs to be achieved. So, here we know that this is the desired level of return or target return and this is the return earned on individual security where securities could be 1 to n number of securities.

So, the weighted average rate of return for all the securities should be equal to the rate of return that the investor is hoping to earn by investing in the portfolio of n assets.

Now, with this we have one more constraint here which suggest that the sum of all the weights which is basically the proportion of fund invested in all the assets right from 1, 2, 3 till n, it should be 1 which means there is no short selling allowed or there is no short selling constraint. Essentially, it shows that the total sum of the weight invested in all assets, weights of the fund or weights of the portfolio assets should be equal to 1.

So, this is the fundamental n asset Markowitz portfolio problem which where we try to achieve the minimum level of risk for the portfolio given a desired level of return and what will be the another constraint. Basically, the next other constraint is the sum total of total investment made in all individual securities should be 1.

So, basically if we try to visualize it, we know that on a two-dimensional graph if we have risk and return and we know that we have some sort of efficient frontier, where we try to have one risk free asset and risky portfolio. What we want to achieve here, what we want to do here is we want to try to find a portfolio where we can invest and achieve this desired return and try to minimize the portfolio risk as well. So, this is the point that this objective function is trying to explore.

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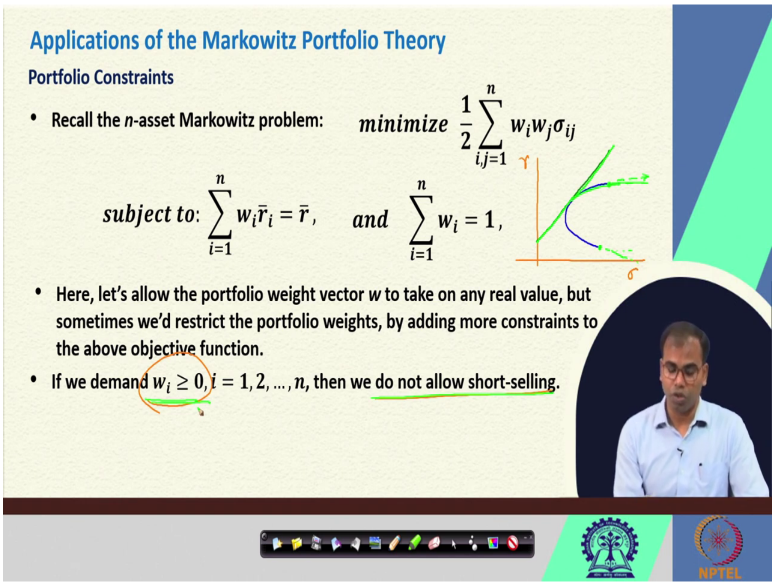

Now, with this basic n asset Markowitz portfolio problem, we move further and try to solve it first. To solve it, let us allow the portfolio weight vector that is w, essentially the vector of all the weights to take any real value and sometimes let us remember that we restrict the portfolio weights by adding more constraint to the above objective function. As I was talking about the short selling constraint, where we might not like to short sell or leverage the portfolio as discussed earlier.

With this if we demand weight of individual assets; individual security is typically non-zero which is most likely positive. And, these individual weights could be ranging from asset i which is basically asset 1, 2, 3 till n. And, if we follow this constraint, we essentially do not allow short selling which means the individual asset will not be purchased by short selling another asset.

Again, in order to visualize the same situation where if we have risk and return; then we know that we are talking about a portfolio efficient frontier, where we have risk free rate. And, we know that we are not short selling, if we short sell then our efficient frontier will be moving from here to here beyond this point or it can go beyond this point also.

So, effectively it will not move, it will not be restricted to this point only, it will go beyond this point if we are short selling or in other case also if we are short selling, it could be beyond this point as well. But here if we use this constraint weight being greater than equal to 0, then we are not allowing short selling.

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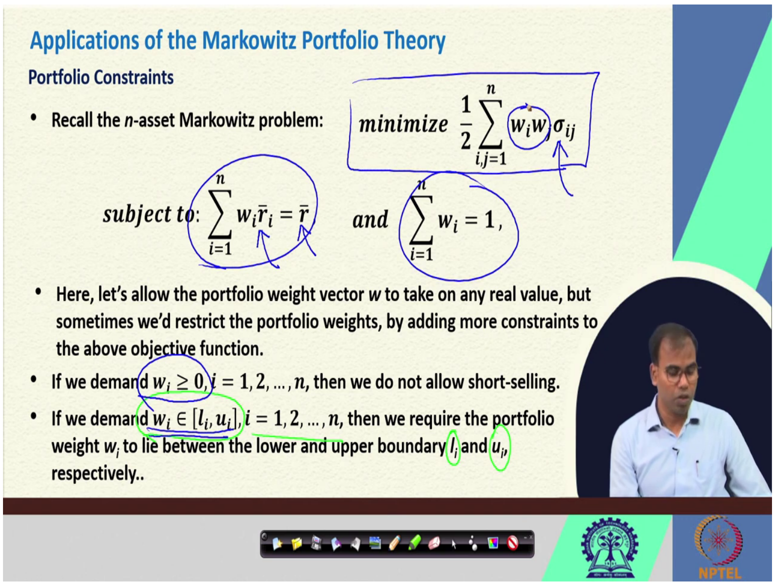

Now, with this situation we need to also put another demand that is basically the portfolio weights w i which is the weight of individual security to lie between the lower boundary l i and the upper boundary. So, essentially, we are trying to restrict the weight of the portfolio to be within a particular range for all the assets that are included in the portfolio.

So, once we have this set these two conditions, where weight should not be less than 0 and the weight should be within a particular range of lower boundary and upper boundary l i and u i respectively. Now, with this these two additional constraint and the original constraint that we already have in the original and the basic n asset Markowitz portfolio problem, we need to find the solution for this which will be given in terms of the value for w i and j.

Let us remember that we are given more most of these inputs. For example, we know the sigma of individual securities, we know the individual risk. We know the rate of return of individual securities; we also know the expected rate of return from the portfolio. And with this given information, we want to find out the securities, the weight of individual securities in which proportion we have to invest in those securities to make the portfolio.

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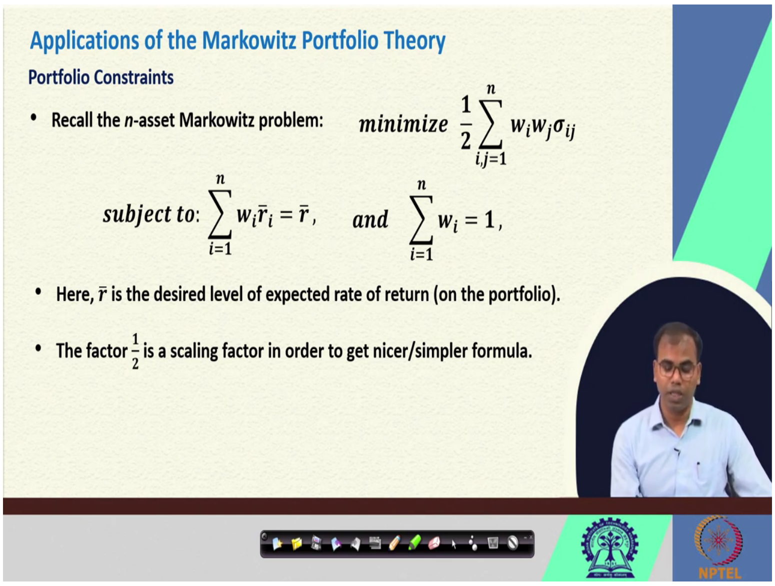

With these constraints, if we go further as already mentioned r dash is basically the desired level of expected return on the portfolio which means the target return or desired return from the investor side and the factor one point 1 by 2 is the scaling factor so, that we can get nicer, simpler formula at the end of the day.

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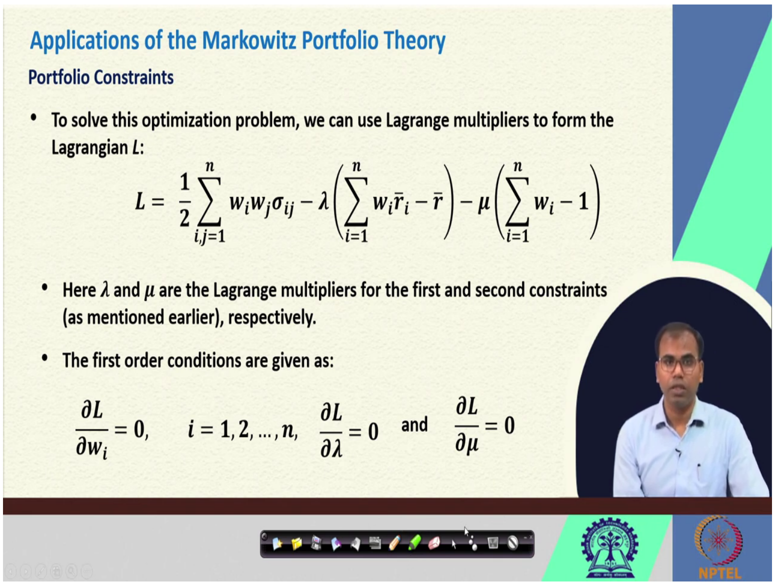

And, with this objective function and constraint as defined earlier, we will solve this optimization problem by using the Lagrangian, Lagrangian multiplier to form the Lagrangian L.

Now, the Lagrangian L would look like something of this sort, where we have these ah factors such as weight of individual security, the variance covariance of individual securities, the rate of return of individual securities, the rate of desired target return. And, with these factor in addition to these factors, we have two more inputs in this Lagrange which is basically lambda and mu.

Now, with these two additional factors, essentially these two factors are the Lagrangian multipliers for the first and second constraint as mentioned earlier, where we have to have the rate of return to be equal to the desired or target level of rate of return. And, we should always have all the weights put together to be equal to 1. With these two constraints, we entered this Lagrangian multipliers and we try to find the first order condition as given earlier as differential of Lagrangian L with respect to weight.

So, d differential of L divided by dw i is equal to 1, where i stands for all the individual assets starting from 1 to till n. And, similarly we have differential first order differential condition of Lagrangian with respect to mu which is basically the first condition is equal to 0.

And, similarly for second condition differential of L with respect to mu, that is the sum total of all the condition representing the sum total of all the weights to be equal to 1 and if this given as 0. So, we set the first order condition in the following manner and then we solve this Lagrangian.

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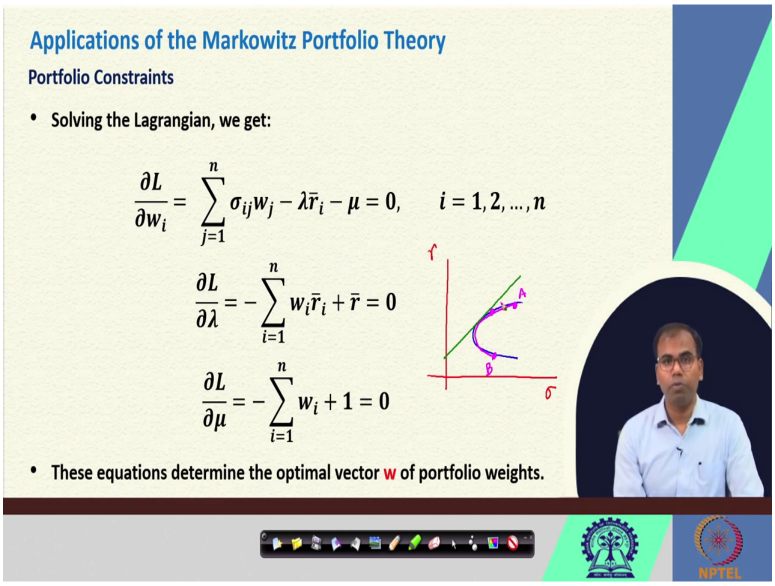

Now, once we solve this Lagrangian, we get the following value. If we look at this, we have this differential of Lagrangian with respect to weight as a function of the individual variance covariance weight of the asset.

And, then we have the rate of return for individual securities and the conditions and subsequently we try to find the solution for all these conditions, first and second conditions with respect to the sum total of the returns being weighted average rate of returns being equal to the return that the investor desires or the target return.

And, the sum total of all the weights in invest or the proportion of money invested in individual assets to be put together equal to 1. With this the three equations essentially determine the optimal vector w of portfolio weight, portfolio weights basically. So, we try to find as started with, we know that the job of this particular Markowitz portfolio problem is to find the amount or the proportion of money that we need to invest in two or more assets or n assets in order to maximize the return or minimize the risk for a given level of return.

So, again going to the visual representation, what we are trying to do is we have this risk return two-dimensional graph where we have this portfolio efficient frontier and we have this risk-free rate. So, what we try to do here is we are trying to find the combination of two assets such that we have the maximum return. So, if we let us say I have two assets A and B and these two assets are lying here and here.

Then, the portfolio frontier here will be of some sort of this curve which will depend on the correlation between the two assets. And, we need to find how much of money to be put in asset A and how much of money to be put in B and that will determine the position of the portfolio on this curve. So, this these equations basically determine the optimal vector w of portfolio weights.

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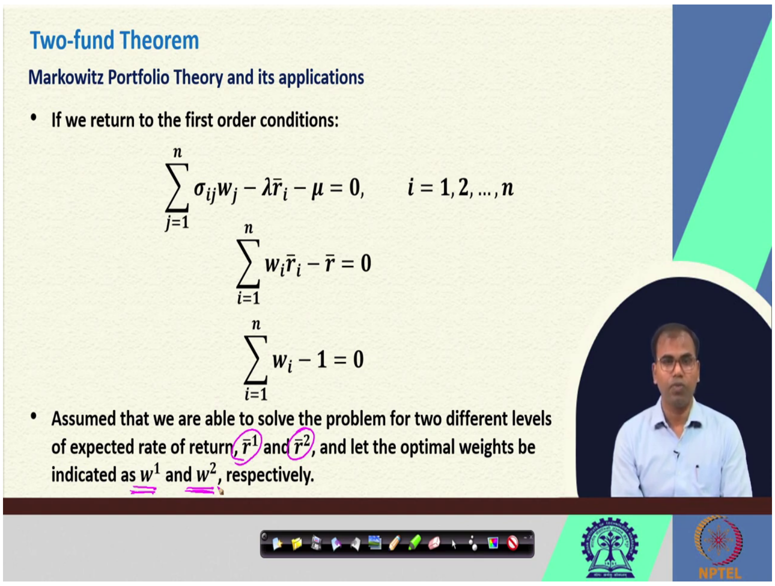

And, once we are able to find this w, we will be able to solve the problem for portfolio choice for any investor. Now, let us try to go back and return to the first order condition. Now, we know that Markovitz portfolio theory has several applications and one of the applications is seen in two fund theorem and to understand in detail the two fund theorem, let us return to the first order condition.

Now, first order condition suggest that the objective function is there, where we will try to find the solution for Lagrangian. And, we have the constraint function as well constraint defined as well where we have to find the target return and the total sum total of the weights of individual securities or the proportion of funds invested in individual securities is equal to 1.

So, the first order condition is given as this and if we further assume that we are able to solve the problem for two different levels of expected rate of return. Earlier, we started with single target return which was r dash. If we assume that we are able to solve this particular function, this particular objective function for two different levels of return; let us say r dash 1 and r dash 2 and this will give us the optimal weight as w 1 and w 2.

Let us assume that for the sake of simplicity, that we are able to find the weights which is assigned to be assigned to the asset 1 and asset 2 as weight 1 and weight 2, w 1 and w 2 respectively.

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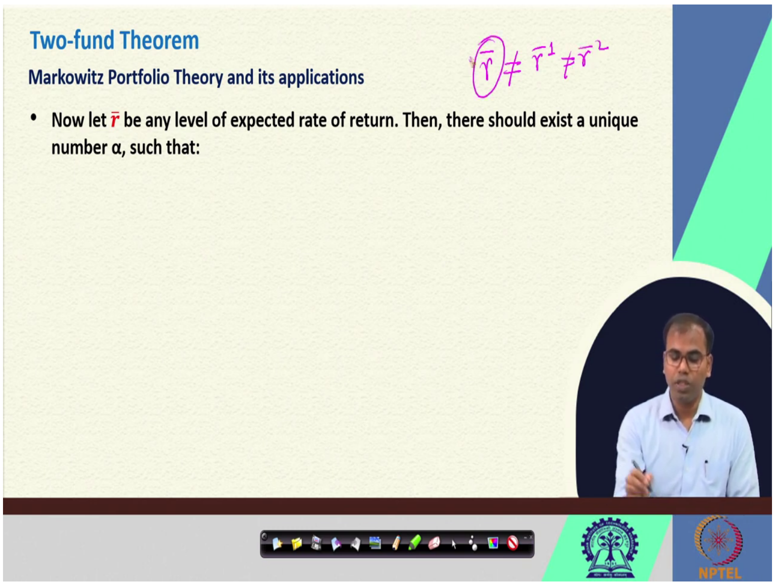

Now, with this additional assumption if we move further, now let us have this new return be any level of expected return for an investor. Now, let us make it clear that this return is not the return that we are already talking about. So, this is a new return. Now, this new return is a desired return that investor wants and it is different from what is already available in the market.

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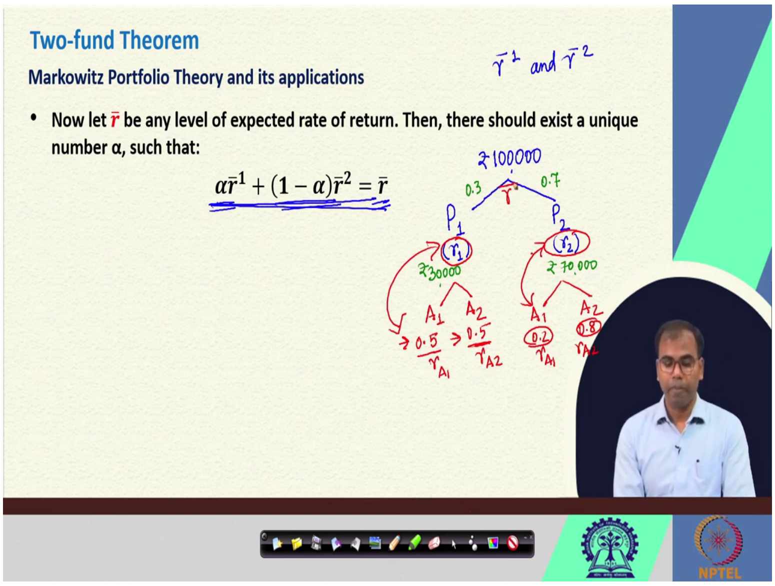

If this is the case, then there should exist a unique number alpha such that, this alpha is a combination of or basically the weight assigned to something that is generated in terms of return from the first iteration of the equation and for the second desired level of return. Let us remember that we have already solved the weight for r 1 and r 2. So, we already know these two returns and the weight that basically is going to help us achieve these two returns.

Now, if we have known this r 1 and r 2, we assign alpha such that the weight of the alpha multiplied with return for 1 and 1 minus alpha with return 2 is equal to the return that the investor now desires. Now, let us try to understand this in a very simple terms. Suppose, an investor has a choice to invest in a portfolio that is combination of some assets and this portfolio yields certain return. Let us call it r 1 and similarly there is another combination of assets as portfolio 2 and this portfolio gives r 2.

Now, as an investor if I want to earn a return, that is different from r 1 and r 2, but I want to still invest in these two assets, these two portfolios only. So, I can invest some proportion of my money in portfolio 1, which will generate a return r 1 and remaining money in portfolio 2 which will generate r 2. So, my total return will be given by the return of as portfolio 1 multiplied with the money that I have invested in portfolio 1.

And, return in from portfolio 2 into the remaining weight or remaining money that I have invested in portfolio 2. So, here this particular equation essentially tells us the same. Now, I give very simple numerical example. Suppose, an investor has 100000 rupees to invest and this investor decides to choose two portfolios. Let us call it portfolio 1 and portfolio 2 which generates return 1 and return 2.

This portfolio 1 is given a weight of let us say 30 percent and remaining 70 percent goes into portfolio 2, which means 30000 rupees will be invested in portfolio 1 and 70000 rupees will be invested in portfolio 2. If we go further, we might be able to figure out that portfolio 1 is a combination of let us say asset 1 and asset 2 in certain proportion, let us say 50 percent and 50 percent. So, we know that this r 1 is a function of r A 1 and r A 2, in proportion of their respective weights.

Similarly, r 2 will be a function of let us say A 1 and A 2 in certain different proportion, let us say 20 percent and 80 percent. So, this will be a different r A 1 and r A 2 combined with their respective weights and thus this r 2 will be determined. Similarly, in r 1 it will be determined on the basis of the weights assigned to individual assets and their respective returns and this will eventually decide the return that investor desires to earn.

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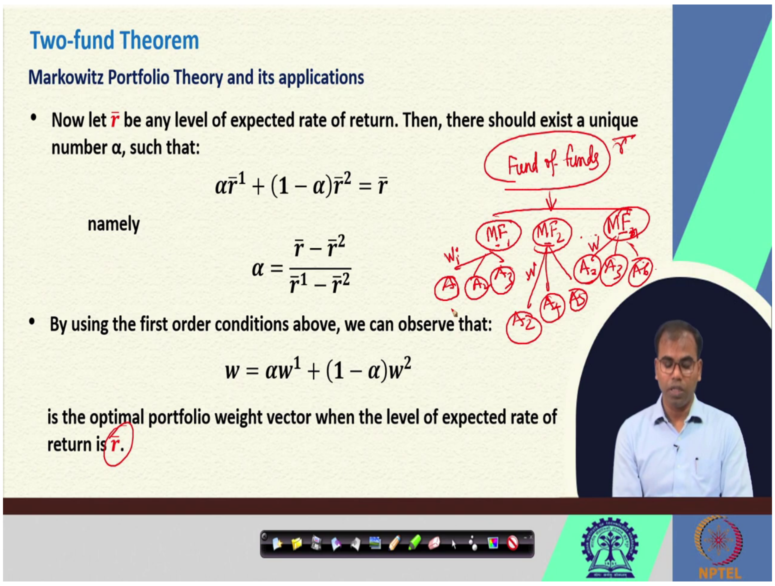

So, with this argument, we know that this alpha can be given as this particular function where we have the target return r dash and then we have r 2 and r 1. And, with this basically using the first order condition for this particular solving this particular problem, we can observe that this w can be given as alpha into w 1 plus 1 minus alpha into w 2.

And, this way we have the optimal weight or optimal portfolio weight vector, when the level of expected return of rate of return is given in terms of r dash. And, then we can figure out what could be the weight to be assigned to portfolio 1 and portfolio 2. And, subsequently it will be leading to the first condition first order condition including the sum total of all the weights put together is equal to 1.

And, the rate of return of individual assets to be invested in is equal to the desired or target level of return. Now, this has implications for mutual fund industry because most mutual funds are designed in such a way that they are invest the mutual funds basically invest in different assets. And, those assets can form the basis of mutual fund and mutual fund can be further added in another portfolio.

So, we have the fund of funds where the return is determined not only by the weight assigned to individual funds within the fund, but also on the individual assets that are part of the first level of funds. So, if we try to understand this in such a way, let us say we have fund of funds where this fund of fund is investing money in mutual fund 1, mutual fund 2, mutual fund 3 and so on. And, this mutual fund is investing in asset 1, asset 2, asset 3 and so on.

This mutual fund 2 is investing in asset 2, asset 4, asset 5 and similarly this is investing in asset 2, asset 3 and so and so on, asset 6 whatever. And, then the these return of these mutual fund will be based on the return of individual assets as part of this mutual fund and the respective weight. Similarly, a return of mutual fund 2 will be dependent on or return on these individual assets and respective weights.

And, similarly for mutual fund 3, the return will be dependent on return on individual assets and with respective weights. And, subsequently the return of this particular fund of funds will be dependent on the money that is invested in MF 1, that is mutual fund 1, that is mutual fund 2, mutual fund 3 and so on. And, the corresponding assets that are that where the mutual fund 1, mutual fund 2 has invested their money respectively and, this way we can create as many funds as possible.

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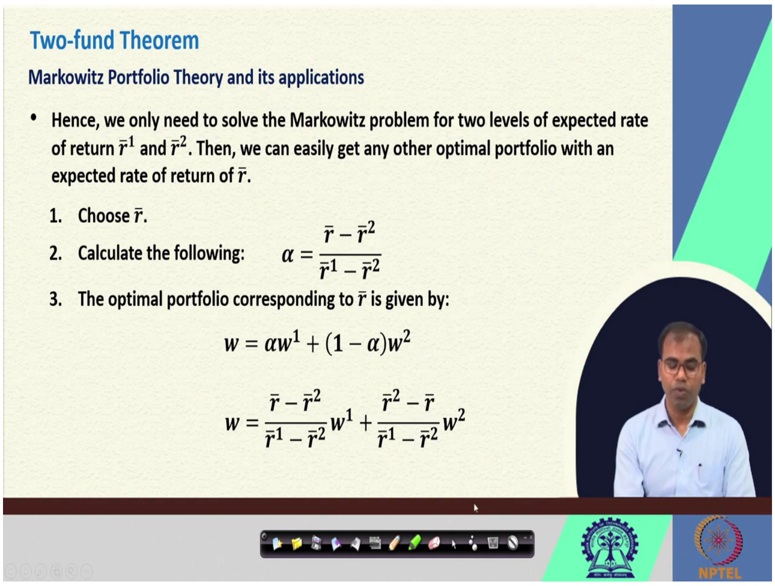

And that is what the whole idea of having this two fund theorem in practice, where we have we need to solve only the Markowitz problem first two level of expected rate of return that is r dash 1 and r dash 2. And, then we can easily get any other optimal portfolio with an expected rate of return of r dash. The step is first we have to decide what target return we want to earn; we want to achieve that is r dash.

And, then we calculate this weight of alpha or this value of alpha using this equation. And, then optimal weight portfolio corresponding to the desired level of return or target return can be given by this formula, that is w is equal to alpha into weight 1 plus 1 minus alpha into weight 2. And, this can be giving us the value of individual weights and value of alpha and 1 minus alpha.

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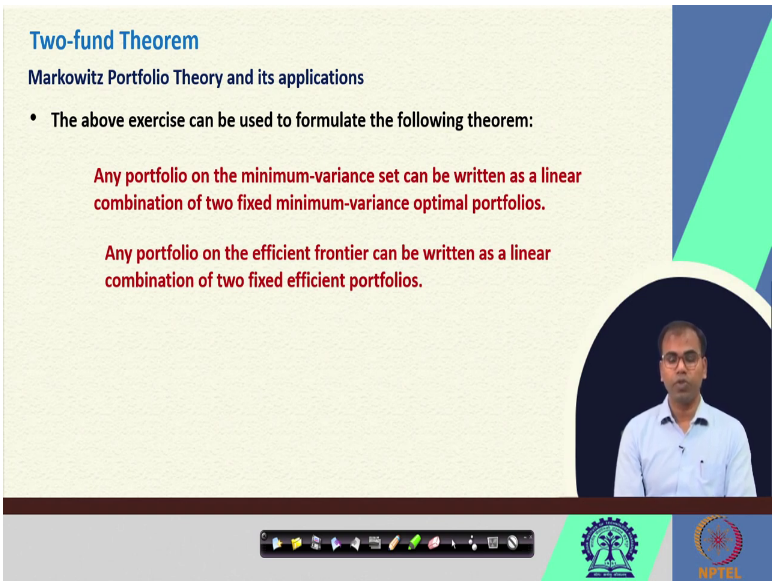

And, this exercise can be used to formulate the following theorem which is known as the two fund theorem which states that any portfolio on the minimum variance set can be written as a linear combination of two fixed minimum variance optimal portfolios.

Or, in another word any portfolio on the efficient frontier can be written as a linear combination of two fixed efficient portfolios. And, this way we can construct as many funds as possible given that we have two funds available and we can create by combining those two funds in different proportion by applying this two fund theorem.

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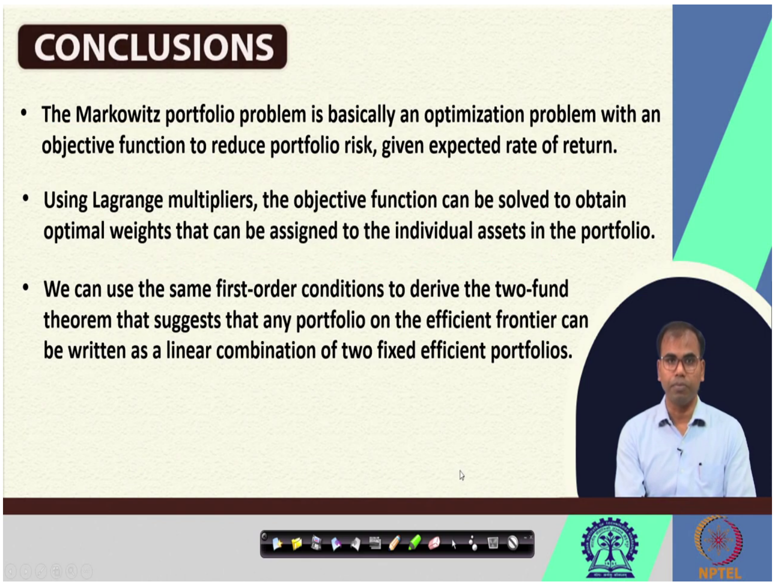

To summarize this, we know that the Markowitz portfolio problem is basically an optimization problem for an investor where investor has an objective function to reduce the portfolio risk, given expected rate of return. We can use Lagrangian multipliers, the objective to solve the objective function and obtain the optimal weights that is basically the proportion of funds invested in individual assets; so, that we can minimize the portfolio risk.

We can also use the same argument of first order condition to derive the two fund theorem, that suggest that any portfolio on efficient frontier can be considered as a linear combination of two fixed efficient portfolios. And, that is the basis of mutual fund industry basically which we will learn in more detail subsequently. For now, that is all.

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Thank you very much.