Applied Econometrics

Prof. Tutan Ahmed
Vinod Gupta School of Management
Indian Institute of Technology - Kharagpur

Module - 3
Lecture - 26
Law of Large Numbers (LLN)

Hello and welcome back to the lecture on Applied Econometrics. So, we have been talking about some natural laws, and in the previous lecture, we have talked about central limit theorem. Now, in today's lecture, we are going to talk about another important natural law, which is law of large number or we sometimes call it LLN, law of large number in short.

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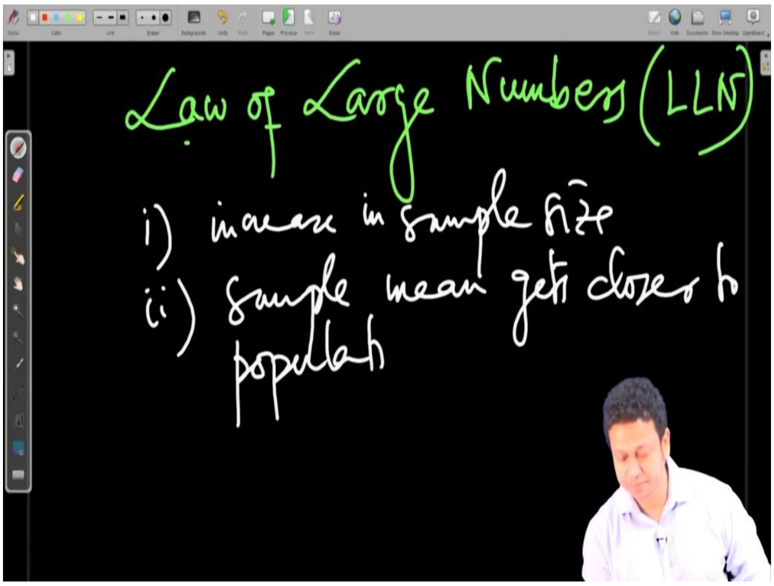

So, the law says, if we keep on increasing the sample size, the sample mean will get closer to the population mean. So, increasing the sample size, the sample mean gets closer to population mean. So, that is the statement. Now, if we express this law, the main statement of the law mathematically, I can say that;

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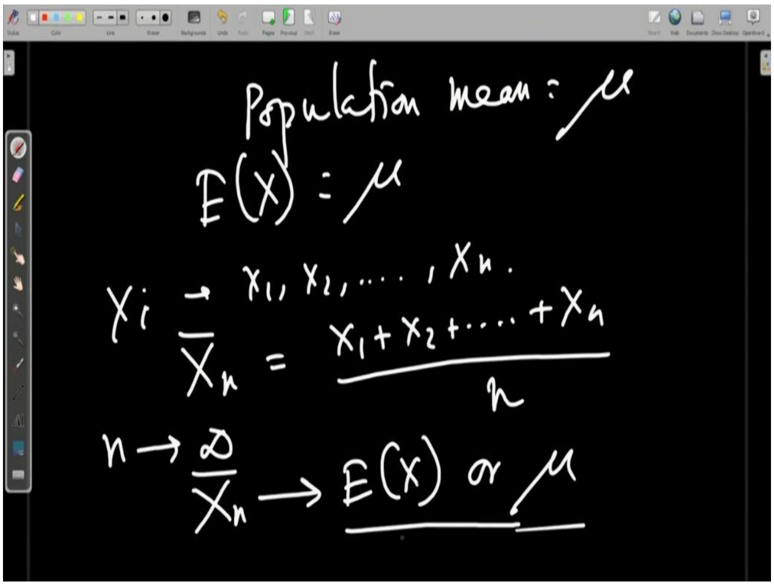

Let us say, the population mean which is of course a constant; population mean is equal to, let us say, some mu, and which is nothing but the expected value of the random variable; so, and the random variable represents all the values of the observations in the population. Now, I can write; so, each of the observation X i; and X i is essentially nothing but X 1, X 2 for the different samples, X n.

Now, if I get a mean of all the X i's, so, I will get X 1, X 2 ... X n divided by n. Now, as n tends to infinity, the law says that X n will also tend to expectation of X or mu. So, the mean that you get from your sample, would tend to the population mean. So, that is the statement. Now, how do we intuitively make sense of this natural law? So, let me again draw; let us say there is a real sort of population.

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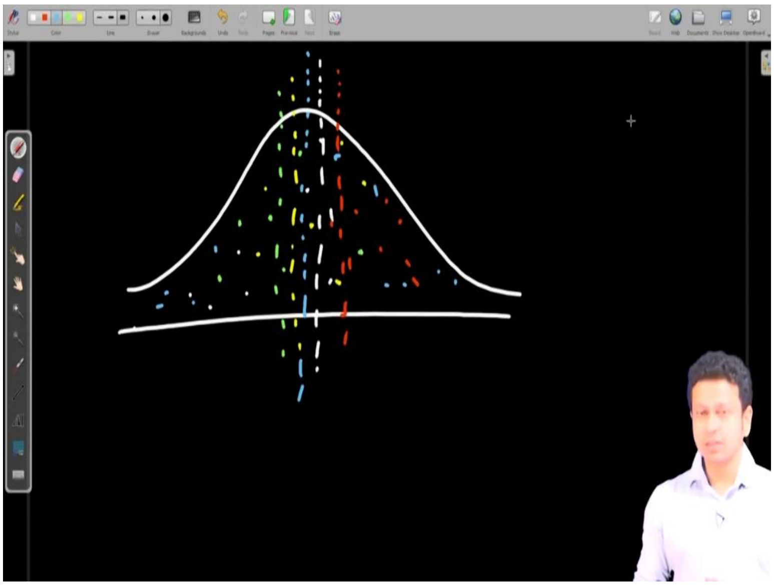

I will just draw some sort of a bell shaped curve. Let us say this is the origin or sort of actual population distribution, and you get some samples from here. Let us say, this, this; first time you get this with these numbers. You get all these values. And corresponding the mean value, you get perhaps somewhere here. Now, second, you add some new values here. And when you add these new values, let us say the mean is going to be somewhere here.

Then you add some more values; let us say there, somewhat here on the reds this side. And the mean is going to be somewhere here. Then we add some more observations. And then the mean is coming here. So, essentially, as you can see, as you can keep on increasing the number, you are covering sort of every part of the distribution. So, every different part of the distributions, you are covering.

And the more number you take, the more the mean will get closer to the population mean. So, with more number of draws, with more number of observations, the mean that we will get, that will be closer to the population mean. So, let me demonstrate this law of large number using Excel sheet.

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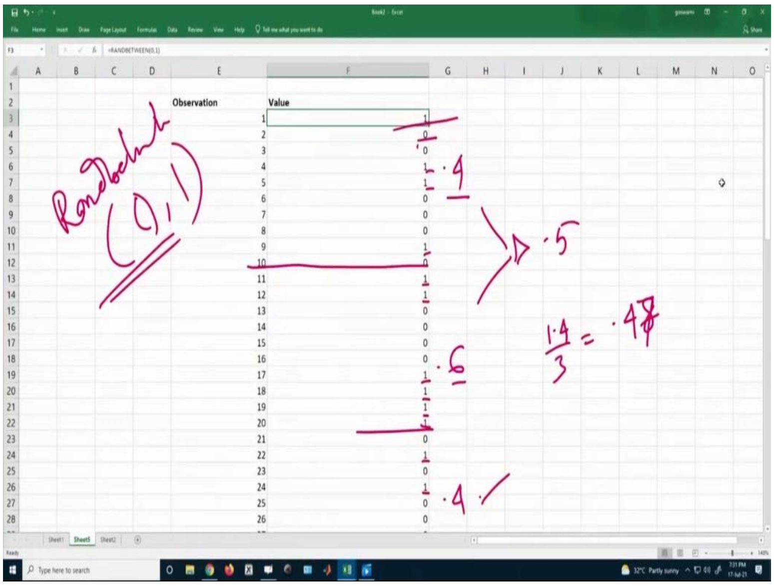

So, we will create a set of values. We have around 70 observations here. We have taken, I have already created it for you. And I have used the function RANDBETWEEN that we have used previously, and I have taken the value RANDBETWEEN 0 to 1. So, that means, it will only create either 0 or 1. So, that is essentially going to be; this is just the same as our Bernoulli trial or a binomial trial.

So, what we will get is that, either the value is going to be a 1 or a 0. So, we can construe 1 to be a success and 0 to be a failure. And let us say I take the first 10 observations. We will see where the mean lies; if I keep on increasing, where the expected value lies. So, if I take the first 10, I will get 1, 2, 3, 4 ones; so, that is 0.4. The probability of success is going to be 0.4, considering 1 to be success. Now, here, from 11 to 20, I get 1, 2, 3, 4, 5, 6; so, I get 0.6. From 21 to 30, I get 1, 2, 3, 4; so, another 0.4.

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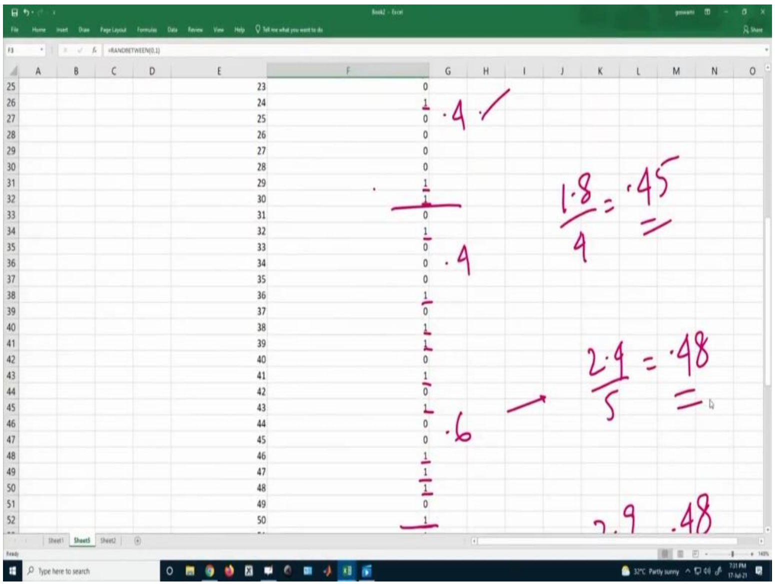

And then, from 31 to 40; 1, 2, 3, 4; so, another 0.4. From 41 to 50; 1, 2, 3, 4, 5, 6; so, this 0.6.

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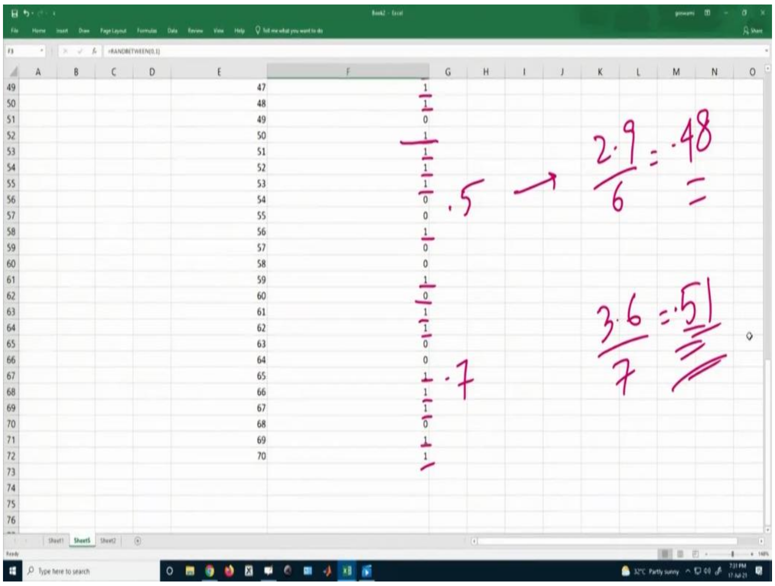

From 51 to 60; 1, 2, 3, 4, 5; 0.5. And from 61 to 70; 1, 2, 3, 4, 5, 6, 7; 0.7. So, let us see what is happening here. So, we are not really coming to be 0.5 in the first few; if I take the mean for the first 10 or first 20, we are not exactly 0.5; but if I take these two, first 20 observations, so, I get something like 0.5, because I take this 0.6 and 0.4. If I take the first 30 observations, so, I get; if I take 0.5 and here 0.4, so, I altogether get 0.1, 0.6, 1.4 by 3, which is going to be 0.47.

Now, if I add another 0.4, so, I will have 1.8 by 4, which is going to be 0.45. Then, if I add 0.6 here, so, I will get 2.4 by 5; so, that is going to give me 0.48. If I add 0.5 here, so, we are going to get 2.9 by 6; so, that is going to give me 0.48 again. If I add this 0.7 here, I get 3.6 by 7; so, it is 0.51. So, we will see, essentially, what we see is, as we sort of keep on increasing; So, initially, I got from from the first trial, it was 0.4.

If I take first two, I get 0.5; and then I get 0.47; then I get 0.45; then I get 0.48; then, again I get 0.48; then I get 0.51. And if we keep on increasing, we will see that it is coming to be sort of stable around 0.5. So, that is basically the essence of the law of large number. Now, I will go back to the whiteboard and I will actually; there is just a similar way, a Bernoulli trial. So, there are many Bernoullis by the way. So, Bernoullis were a very famous family and they are all mathematicians, and they all contributed significantly; so, I do not know which Bernoulli this one is.

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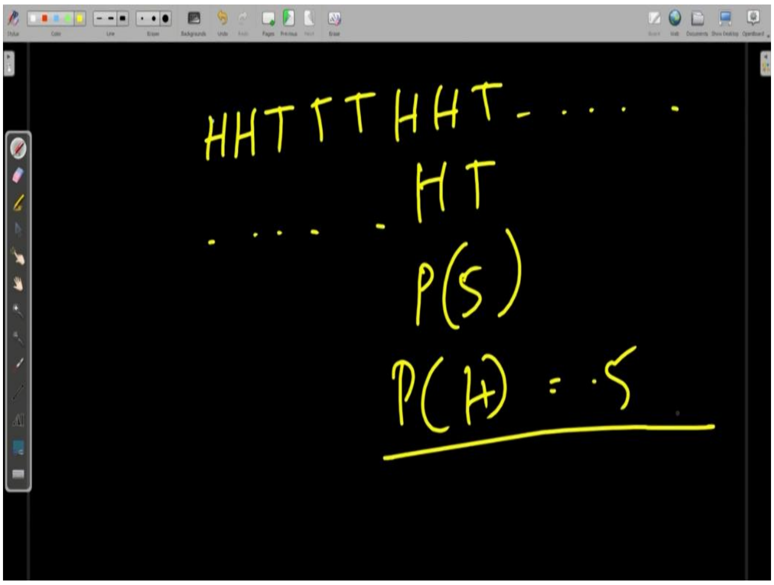

There was a Bernoulli trial where that Bernoulli actually saw, he repeatedly tossed coins and he got the results; and it was like for a long chain of experiment. When he did that, it was found that, at the end of the day, the probability of success is going to be, your probability of head is going to be 0.5. So, it is like sort of obvious that if you keep on repeating the experiment, if you have very large number of observations, what we are going to get is that the probability of success to be the true representation of the population.

So, that is basically the idea. Now, how do we mathematically prove? So, we want to see some mathematical proof why it is happening; and we can actually do it. So, similar to the previous one, if I have, let us say, if I have all my samples, say X 1, X 2, X 3 ... X n.

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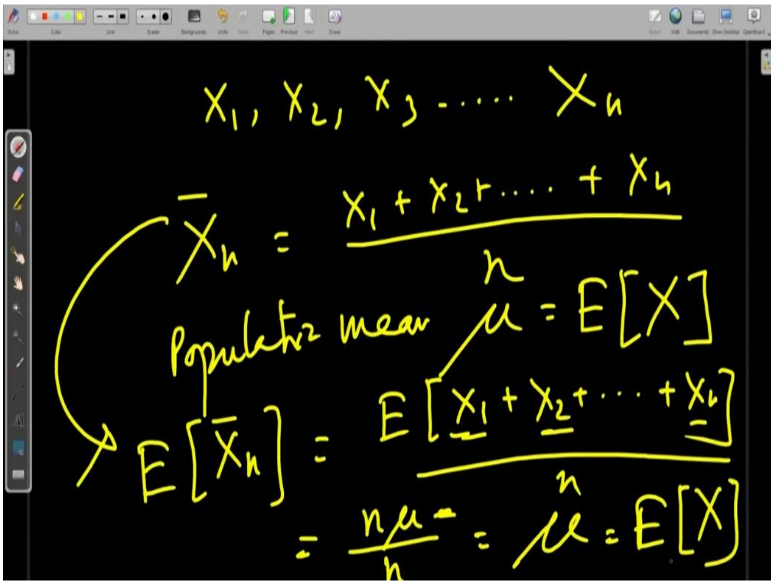

And if I want to get a mean of X n, and which is basically; X 2; X n by n; so, that is going to be my; and my population mean mu is equal to expected value of X. So, if I sort of take expected value of X n, if I take expectation of this one, so, then I will get expectation of all the X 1, X 2 ... X n by n. So, essentially, it means I will get; so, expectation of all these different samples is going to give me n mu by n, because all expectations are n, and I add them up; so, I get n mu.

So, this is going to give me, is equal to mu, which is nothing but the population mean expectation of X. Now, what about the variance?

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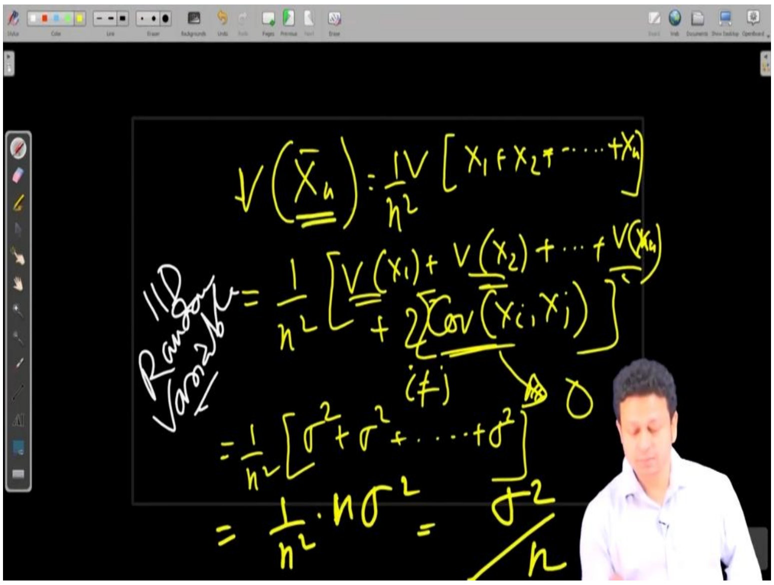

If I take a variance of the previous one, the variance of this one, so, I get V X n bar is basically V X 1, X 2 up to X n by; I have to take the n square out; so, for variance, it is going to be 1 by n square. And if I take the variance, so, I will have V of X 1, V of X 2 ... V of X n. And then you will have all this covariance terms, covariance X i, X j's, where i is not equal to j. Now, this covariance terms will be 0, because we are considering IID random variable here also.

So, when the covariance terms are 0, so, what I will be left with is 1 by n square. And since this is IID random variable, all the variances are going to be sigma square; sigma square plus sigma square plus ... sigma square. Now, if I add this up, so, I will have 1 by n square into n sigma square; so, which is nothing but sigma square by n. So, that is going to be the variance of the X n bar that we have drawn. Now, if I now want to actually prove the law of large numbers, so, I am going to use Chebyshev's inequality.

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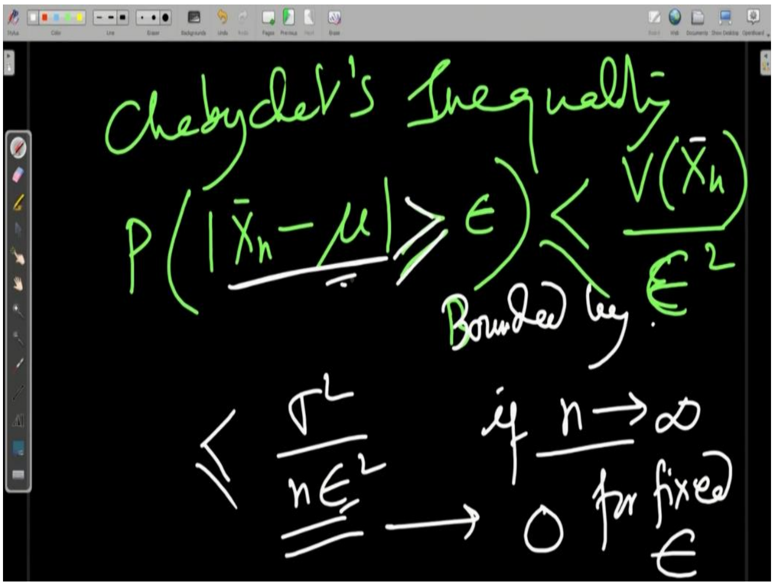

So, what Chebyshev's inequality is saying is that, probability that this X n or X n bar, if you want to write, is away from the mean if it has to have the probability of this X n bar to be away from the mean; if we keep, it has to be greater than some fixed Epsilon value. Then it is going to follow this; Epsilon square. This is going to follow this. This is basically bounded by this.

Now, if I substitute the variance of X n bar, so, I am going to write, X n bar is equal to sigma square by n Epsilon square. Now, if n goes to infinity, so, this value is going to go to 0, for a fixed Epsilon. Now, this is basically the idea. So, if you have very large n, so, the probability that X n bar is away from mu is, with a value higher than Epsilon is actually 0. So, that means that X n bar is going to be pretty close to my mu.

So, that is basically the idea of law of large number. Now, we have explained; so, this is basically the mathematical explanation and we have seen from our Excel sheet or we have actually sort of got some intuitive understanding of law of large number. Now, again, we will try to see, we will try to make some intuitive understanding as in why it is happening. And again, we will go back to Mr. Galton.

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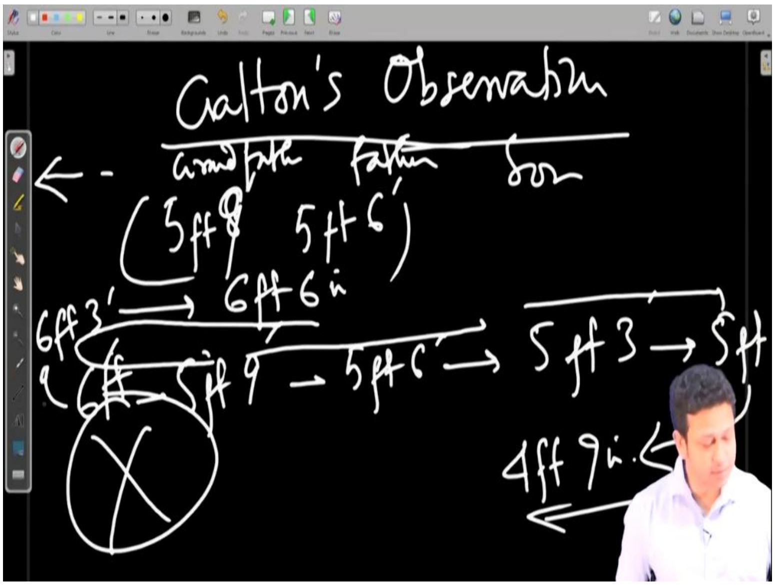

Mr. Galton's observation; so, the same Francis Galton about whom we talked about in central limit theorem. So, Mr. Galton, this time, what he did was, he actually got a lot of details of height across the generation. So, let us say father's height is 5 feet 6 inch; grandfather's, let us say 5 feet 9 inch and so forth. And he has got these different lineages; so, say if grandfather, father, son and so forth.

So, now, the idea is, before I get into the observations made by Galton, let me ask you a question. Now, let us say, if the father's height is 5 feet 6 inch and the grandfather is 5 feet 9 inch, how likely is his father was 6 feet, or his father was 6 feet 3 inch; or this person's son is going to be 5 feet 3 inch; or further, if the next generation is going to be 5 feet; or going forward, if it is going to be 4 feet 9 inch; so, how likely is that going to happen; or this person's father is 6 feet 6 inch.

So, how likely we are going to see this kind of situation? So, it is unlikely to happen, because, it cannot happen that it automatically goes on decreasing, and it is sort of diminishing across the generation.

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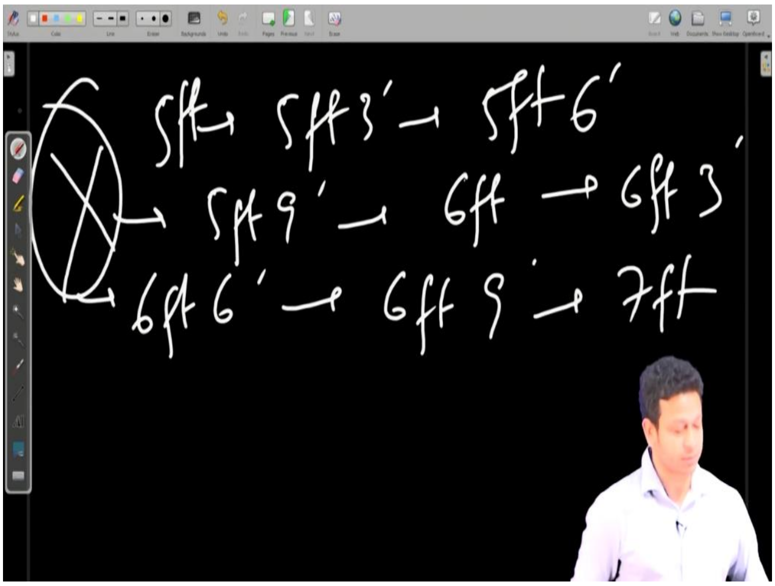

Or, on the other side, if some someone's father is 5 feet, it is not likely his son is going to be 5 feet 3 inch, 5 feet 6; I mean, across generation, it is not going to be this; 5 feet 9 inch, 6 feet, 6 feet 3 inch, 6 feet 6 inch, 6 feet 9 inch, 7 feet. We are not going to see this kind of situation. We are not going to see this kind of situation, but instead, what we are going to see;

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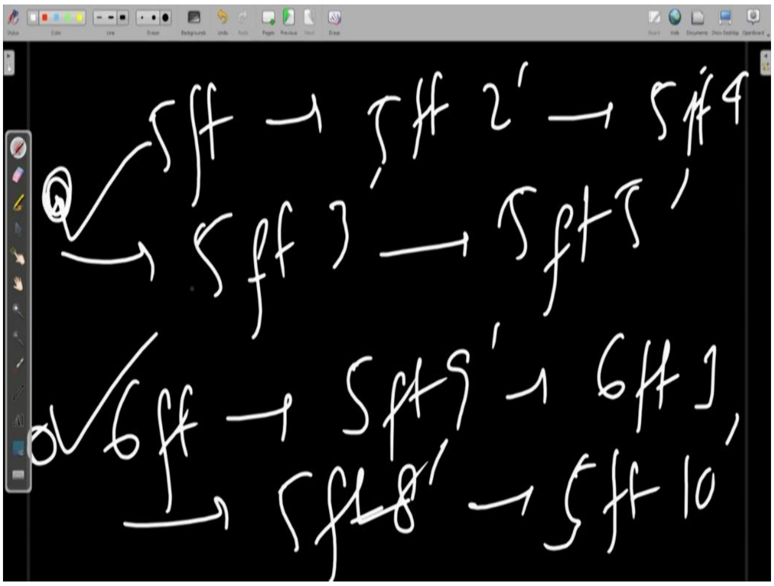

Let us say, if the father is 5 feet, maybe the son is 5 feet 2 inch; his son is 5 feet 4 inch; and then, again it comes back to 5 feet 3 inch; and then 5 feet 5 inch; something like that. It will not go in one way or the other. Similarly, for the 6 feet farther, it is not likely, it might be 5 feet 9 inch; then it is going to be 6 feet 1 inch; then it is going to be 5 feet 8 inch; and then it is going to be 5 feet 10 inch and so forth.

So, it will somewhat come towards some mean value; it is not going to go in one way or the other. So, basically, and this is the observation exactly; this is the same observation Galton made.

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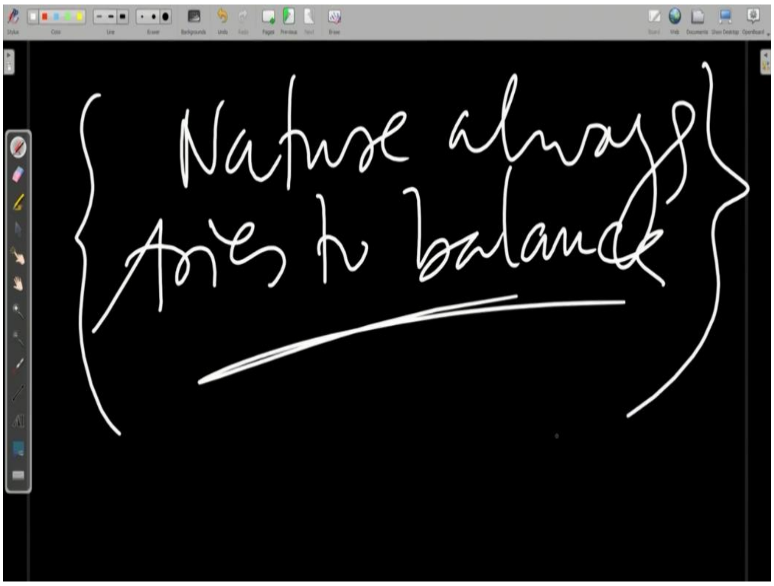

So, basically, the point is that, nature always tries to balance. So, if one generation has more height, the next generation might have a less height; and the next generation, maybe even less height; and then again, it will go up. So, this way, nature will always try to create a balance. So, we will always see, when we take lots and lots and lots of observation, it is always going to come to some natural sort of height.

So, a human height may be, a natural, normal human height, if you take the mean of the entire population in this world, so, it is going to be 5 feet, maybe, I do not know, 6 inch, 7 inch, something like that. So, that is basically the thing. So, always, nature is trying to balance, and that is the crux of the law of large number. Now, I will end this lecture with some philosophical note.

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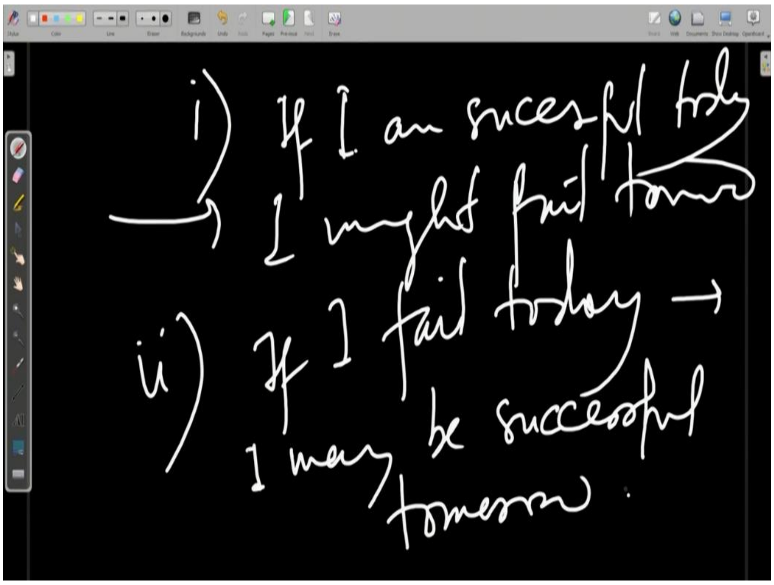

This law of large number has a really important philosophical conversation, and that is, it means, if I am successful today, according to the law of large number, there is a high chance that I might fail tomorrow, because nature always tries to balance. Or if I fail today, there is a high chance that my next attempt, I am going to be successful, I may be successful tomorrow. And things are always going to be stable in the long run.

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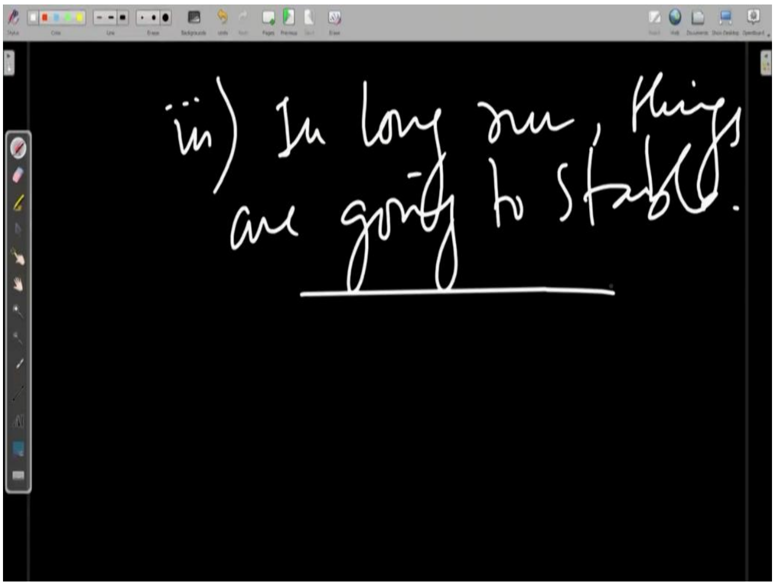

In long run, things are going to be stable. So, if you are successful today, you need to be humble that, it teaches you humility that there is a chance that you might fail; or if your failure today, it inspires you to feel that you are going to be successful tomorrow. And you should not always, like; because things are going to be stable in the long run, there is nothing to worry about; things are going to be fine; so, it is according to the law of large number.

So, with this note, we will end this lecture. And in the next few lectures, we are actually going to see some really important probability distributions that we are going to use in our day to day life and other problem solving.